Non Cooperative Household Demand

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Abstract

We study noncooperative household models with two agents and several voluntarily contributed public goods, deriving the counterpart to the Slutsky matrix and demonstrating the nature of the deviation of its properties from those of a true Slutsky matrix in the unitary model. We provide results characterising both cases in which there are and are not jointly contributed public goods. Demand properties are contrasted with those for collective models and conclusions drawn regarding the possibility of empirically testing the collective model against noncooperative alternatives and the noncooperative model against a general alternative.

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1 Introduction

Demand analysis has never been more important for policy analysis, where it is the key ingredient for a number of policy relevant issues, such as the structure of optimal taxation, or the analysis of demand for the study of industrial organisation and competition policy and indeed of labour supply. However, it has now long been observed that the standard unitary model often leads to estimated demand functions that are problematic. One key reason for this is that the attempt to represent aggregate household demands as resulting from one "representative" optimising household decision maker may generate demand functions incompatible with the implications of models with multiple decision makers.

Indeed Browning and Chiappori [10] have shown that if the efficient collective model is true the Slutsky substitution matrix will generally not be symmetric, but can be represented by a matrix that is the sum of a symmetric matrix plus a matrix of a rank determined by the number of household members. Both in the context of demand analysis and in labour supply there has been a detailed examination of the theoretical and empirical implications of the efficient collective model, as detailed below. However, there are good reasons to believe that efficiency may not hold, if anything because of the informational requirements and the resulting cost of implementing efficiency. Thus in this paper we consider the structure of household demands under the assumption that these are the result of non-cooperative interaction between household members. This allows us to relax the assumption of efficiency made in the collective models as well as the restrictive structure of the unitary model which models household behaviour as if there was one individual deciding.

Maximisation of utility by a single consumer subject to a linear budget constraint implies strong testable restrictions on the properties of demand functions. Empirical applications to data on households often reject these restrictions. In particular, such data frequently show a failure of Slutsky symmetry - the restriction of symmetry on the matrix of compensated price responses (see for example Deaton [19], Browning and Meghir [13], Banks, Blundell and Lewbel [2] and Browning and Chiappori [10]).

From the theoretical point of view, the inadequacy of the single consumer model as a description of decision making for households with more than one member has also long been recognised. Attempts to reconcile this model with the existence of several sets of individual preferences have been made for instance by Samuelson [30] and Becker [3], [4] but rely upon restrictive assumptions about preferences or within-household decision mechanisms (see Bergstrom [5], Cornes and Silva [17]).

A large body of recent research has investigated models accommodating alternative descriptions of within-household decision-making processes. Efficiency of household decisions holds in a number of models of household behaviour which have been suggested: for instance in the Nash bargaining models of Manser and Brown [26], McElroy and Horney [29] and McElroy

[28], and in Browning, Bourguignon, Chiappori and Lechene [9] and Bourguignon and Chiappori [8]. However, it is not a property of noncooperative models such as those of Ulph [32], [33] and Chen and Woolley [14].

An important advance is made by Browning and Chiappori [10], who show that under the sole assumption of efficient within-household decision making, the counterpart to the Slutsky matrix for demands from a k member household is the sum of a symmetric matrix and a matrix of rank $k-1$. Chiappori and Ekeland [15] establish not only that efficiency implies a rank $k-1$ deviation but also that a rank $k-1$ deviation implies the existence of preferences compatible with efficient behaviour. Chiappori and Ekeland [16] show that for these preferences to be identified it is required to know which goods are private and which are public and that it is sufficient for identification to assume the existence of exclusive goods. Browning and Chiappori [10] report tests on Canadian data which reject symmetry for couples, but not for single individual households. The hypothesis that the departure from symmetry for the sample of couples has rank 1, as implied by the assumption of efficiency, is also not rejected.

These results not only fill a gap in our theoretical understanding of demand behaviour but also open the prospect of reconciling demand theory and data on consumer behaviour. The work of Browning and Chiappori [10] and Chiappori and Ekeland [15] is important in showing that the assumption of efficiency generates testable restrictions on household demand functions, clearly distinguishing the collective model from both the unitary and the entirely unrestricted case.

In this paper we explore the same question of the testable restrictions implied by an alternative structural assumption on within household interactions. The model considered is the principal alternative to both the unitary and collective models, that of noncooperative demand behaviour with voluntarily contributed public goods. This model warrants attention in its own right as the only currently widely discussed alternative to fully efficient models of the sort described above[1]. It is also interesting in so far as the equilibria in this model can be considered as the fallback position in bargaining models as suggested, for example, in Woolley [35], Lundberg and Pollak [25] and Chen and Woolley [14].

Models of voluntarily contributed public goods have relevance beyond analysis of household demand. When they involve more than two players, these models can be used to represent a variety of situations involving private contributions to public goods either in the national or international context. What distinguishes what we have termed the "household Nash equilibrium model" from the general Nash equilibrium model is the small number of agents, which is two in the case considered here.

In section 2 we lay out the general framework. The model has two types of equilibria,

[1]D'Aspremont and Dos Santos Ferreira [18] provide an interesting recent attempt to parametrise cases intermediate between fully cooperative and fully noncooperative within-household behaviour.

depending upon whether partners do or do not contribute jointly to a common set of public goods. In section 3, we consider the case in which there are jointly contributed public goods. We show that equilibrium quantities vary with prices and household income in ways compatible with the adding up and homogeneity properties of unitary demands and that negativity and symmetry properties will generally be violated, as in the collective model. We derive the counterpart to the Slutsky matrix, and show that it can be decomposed into the sum of a symmetric matrix and another matrix whose rank generally exceeds the deviation to be expected in a collective setting. Section 4 is devoted to the properties of demands in the case of no jointly contributed public goods. Adding up holds, homogeneity may fail and the rank of the departure from negativity and symmetry is shown to be similar in rank to that when public goods are jointly contributed. Section 5 offers an example. Section 6 discusses how to make use empirically of the result and how to combine it with previous results. In particular, we establish the numbers of public and private goods required for the result to constitute a testable restriction on behaviour. The results suggest that the properties of the Slutsky matrix provide a basis for testing not only the Browning-Chiappori assumption of efficiency but also other models within the class of those based on individual optimisation. Section 7 concludes.

2 The general model

Consider a household with two individuals, A and B. The household spends on a set of m private goods $q \in \mathbb{R}^m_+$ and n public goods $Q \in \mathbb{R}^n_+$. The quantities of private goods purchased by the individuals are q^A and q^B with total household quantities $q \equiv q^A + q^B$. The quantities of public goods purchased by the household are Q with individual contributions Q^A and Q^B and $Q \equiv Q^A + Q^B$. Individuals have smooth preferences represented by utility functions $u^A(q^A, Q)$ and $u^B(q^B, Q)$, increasing and differentiable in all arguments, so that individual preferences are defined over the sum of contributions to the public goods.[2] The partners have incomes of $y^A \in \mathbb{R}_+$ and $y^B \in \mathbb{R}_+$. Household income is denoted $y \equiv y^A + y^B$. Prices of private and public goods respectively are the vectors $p \in \mathbb{R}^m_+$ and $P \in \mathbb{R}^n_+$.

Each person decides on the purchases made from their income so as to maximise their utility subject to the spending decisions of their partner. We can write the agents' problems as

$$\max_{q^A, Q^A} u^A(q^A, Q) \;\text{ s.t. }\; p'q^A + P'Q^A \leq y^A,\; Q^A \geq 0,\; q^A \geq 0$$

and

$$\max_{q^B, Q^B} u^B(q^B, Q) \;\text{ s.t. }\; p'q^B + P'Q^B \leq y^B,\; Q^B \geq 0,\; q^B \geq 0$$

where inequalities should be read where appropriate as applying to each element of the relevant vector.

[2] A good is public in the context of the household if more than one household member cares about it.

A *household Nash equilibrium* consists of a set of quantities (q^A, q^B, Q^A, Q^B) simultaneously solving these two problems. The existence of at least one such equilibrium is established in Browning, Chiappori and Lechene [12]. The equilibrium need not be unique, though Bergstrom, Blume and Varian [6], [7], Fraser [20] and Lechene and Preston [24] provide sufficient conditions, essentially involving normality of both public and private goods, for uniqueness of certain sorts of equilibria.

In any equilibrium, public goods can be divided into two types - those to which only one partner contributes and those to which both do. We refer to the former as *individually contributed public goods*, and denote the quantity vectors for such goods[3] Q_A and Q_B, the respective prices P_A and P_B and their dimensions n_A and n_B. Without loss of generality we assume $n_A \geq n_B$. The latter type, on the other hand, are referred to as *jointly contributed public goods*, with quantity vector denoted X, prices R and dimension n_X. Individual contributions to these public goods are denoted X^A and X^B.

Equilibria can be distinguished into those in which there are and are not jointly contributed public goods. Those in which $n_X \geq 1$ are called, for reasons which will become apparent, *income pooling equilibria* and those in which $n_X = 0$ are called *separate spheres equilibria*.

It is useful to recognise that the problems can be rewritten to have partners effectively choosing the levels of the public goods for the household, subject to the constraint that this level is greater than or equal to the contribution of the other agent. Given that $y^A = y - p'q^B - P'Q^B$, and similarly for B, the agents'problems can be re-written as:

$$\max_{q^A, Q} u^A(q^A, Q) \text{ s.t. } p'q^A + P'Q \leq y - p'q^B,\ Q \geq Q^B,\ q^A \geq 0$$

and

$$\max_{q^B, Q} u^B(q^B, Q) \text{ s.t. } p'q^B + P'Q \leq y - p'q^A,\ Q \geq Q^A\ q^B \geq 0.$$

3 Income pooling equilibria

3.1 Income pooling

In an income pooling equilibrium, the solution to each partner's problem coincides with that to the following problems[4]

$$\begin{aligned} \max_{q^A, Q_A, X} u^A(q^A, Q_A, Q_B, X) \text{ s.t. } p'q^A + P_A'Q_A + R'X &\leq y - p'q^B - P_B'Q_B, \\ Q_A &\geq 0,\ X \geq X^B,\ q^A \geq 0 \end{aligned}$$

[3]Note that subscripts A and B are used to distinguish goods contributed exclusively by individuals A and B whereas superscripts A and B distinguish contributions by individuals A and B (to any good).

[4]For general preferences, there can only be one jointly contributed public good X. We discuss this in section 3.2 below. The exposition here applies to both the general case with $n_X = 1$ and the non-generic cases with $n_X > 1$.

and

$$\max_{q^B, Q_B, X} u^B(q^B, Q_A, Q_B, X) \text{ s.t. } \begin{aligned} p'q^B + P'_B Q_B + R'X &\leq y - p'q^A - P'_A Q_A, \\ Q_B &\geq 0,\ X \geq X^A,\ q^B \geq 0. \end{aligned}$$

Hence, quantities purchased will satisfy

$$q^A = f^A(y - p'q^B - P'_B Q_B, p, P, Q_B) \tag{1}$$

$$Q_A = F^A(y - p'q^B - P'_B Q_B, p, P, Q_B) \tag{2}$$

$$q^B = f^B(y - p'q^A - P'_A Q_A, p, P, Q_A) \tag{3}$$

$$Q_B = F^B(y - p'q^A - P'_A Q_A, p, P, Q_A) \tag{4}$$

and

$$X = G^A(y - p'q^B - P'_B Q_B, p, P, Q_B) \tag{5}$$

$$= G^B(y - p'q^A - P'_A Q_A, p, P, Q_A). \tag{6}$$

where $f^A(.)$, $F^A(.)$, $f^B(.)$, $F^B(.)$, $G^A(.)$, $G^A(.)$ are conditional Marshallian demand functions corresponding to the two partners' preferences and together satisfying the usual demand properties.

We use subscripts to denote derivatives of these demand functions: f^i_y, f^i_p, f^i_P, $f^i_{Q_j}$, F^i_y, F^i_p, F^i_P, $F^i_{Q_j}$ and G^i_y, G^i_p, G^i_P, $G^i_{Q_j}$ for $i = A, B$, with respect to income y, price vectors p and P and individually contributed public goods quantities of the other partner Q_j respectively.

Note that (1) to (4) define $2m + n_A + n_B$ equilibrium equations in $2m + n_A + n_B$ quantities (q^A, Q_A, q^B, Q_B) independently of (5) and (6). Substituting solutions to these equations into (5) or (6) will give the set of income pooling equilibria. Furthermore the set of solutions to these equations plainly depend only upon (y, p, P) and in particular do not depend upon the distribution of income within the household. This well known "income pooling" result is the source of the name given to such equilibria. This result is well known and has been discussed by many authors. Warr [34] established income pooling for the case of a single public good and Kemp [23] extended the claim to the case of multiple public goods, assuming interior equilibrium. Kemp's proof is queried by Bergstrom, Blume and Varian [6] who offer an alternative proof.

3.2 Jointly contributed public goods

Satisfaction of both (5) and(6) with multiple jointly contributed public goods at anything other than isolated values of (y, p, P) clearly requires a certain coincidence in preferences over public goods between the two partners. Browning, Chiappori and Lechene [12] demonstrate that generically $n_X \leq 1$ so that typically there will not exist more than a single jointly contributed public good in equilibrium[5]. More precisely, given a suitable topology on preferences, there is no open set in the space of the couple's preferences, incomes and prices on which $n_X > 1$ in equilibrium. This is not to say, however, that there are not subspaces of preferences within which equilibria with $n_X > 1$ can hold on an open set of values for (y, p, P). What is required is that the partners' marginal rates of substitution between jointly contributed public goods should coincide at all equilibrium quantities of the goods over such a set. That is possible, for example, if preferences over jointly contributed public goods are separable and identical for the two partners. It is, in fact, possible even without such separability if preferences over those individually contributed goods from which there is not separability are also identical between the partners since there exist equilibria with quantities of these goods also identical in equilibrium[6]. Lechene and Preston [24] demonstrate the possibility of such cases. Of course, these cases are not robust to small independent perturbations in the partners' preferences but identity and separability of preferences over subsets of public goods may make sense in certain cases - for example, if the subutility function reflects an agreed technology for producing some intermediate good or if, say, the goods in question relate to children and the subutility reflects an agreed welfare function for the children. We present results covering both the generic case and the possibility that $n_X > 1$.

3.3 Household demands

In what follows we assume uniqueness of the equilibrium and denote the mappings from (y, p, P) to the unique individual equilibrium goods vectors by $\theta^A(y, p, P)$, $\Theta^A(y, p, P)$, $\theta^B(y, p, P)$, $\Theta^B(y, p, P)$ and to the jointly contributed quantities by $\Xi(y, p, P)$. We let

$$\theta(y, p, P) = \theta^A(y, p, P) + \theta^B(y, p, P)$$

[5]Throughout the paper we say that a property holds generically if the closure of the set of couples' preferences, incomes and prices at which it fails to hold has empty interior. To make such a statement we require a topology on smooth preferences such as the compact-open topology discussed in, say, Kannai [22] or Mas-Colell [27]

[6]Specifically, interior equilibria with $n_X > 1$ exist on an open set of values for (y, p, P) if private goods can be partitioned, $q^i = (q_0^i, q_1^i)$, $i = A, B$, in such a way that individual preferences take the weakly separable form

$$u^i(q^i, Q) = \upsilon^i(q_0^i, Q^i, \nu(q_1^i, X)) \qquad i = A, B$$

for some $\upsilon^i(.,.,.)$, $i = A, B$ and some common subutility function $\nu(.,.)$. In such a case, marginal rates of substitution between public goods in X are, for each partner, the same function of quantities q_1^i and X and there exist equilibria with $q_1^A = q_1^B$ so that these marginal rates of substitution coincide as required. Such preferences obviously include, for instance, the cases both of common separability of public goods ($q^i = q_0^i$, $X = Q$) and of identical preferences ($q^i = q_1^i$, $X = Q$).

and

$$\Theta(y,p,P) = \begin{pmatrix} \Theta^A(y,p,P) \\ \Theta^B(y,p,P) \\ \Xi(y,p,P) \end{pmatrix}$$

denote the household private and public goods vectors. Note that quantities are uniquely determined as functions of the same economic determinants y, p and P as would be the case under the "unitary" model where the household maximises a household utility function given the household budget constraint. Distinguishing unitary and noncooperative household behaviour therefore requires that we establish whether these equilibrium quantities have properties dissimilar to demands in unitary households. Browning and Chiappori [10] provide such an analysis for the collective model, and Browning, Chiappori and Lechene [11] examine the relationship between collective and unitary models.

The properties of unitary demands are the standard Hurwicz-Uzawa [21] integrability requirements of adding up (Engel and Cournot aggregation), homogeneity, negativity and symmetry. It is easy to establish that the household Nash equilibrium quantities satisfy adding-up and homogeneity.

Negativity and symmetry are less simply dealt with. These are concerned in the case of the unitary model with the properties of the Slutsky matrix, the matrix of price responses at fixed household utility. Since household utility is undefined in a noncooperative setting, no such matrix is defined but we can adopt the Browning and Chiappori [10] notion of the "pseudo-Slutsky matrix". This in the current context is the matrix

$$\Psi \equiv H \; + \; h \begin{pmatrix} \theta \\ \Theta \end{pmatrix}' \tag{7}$$

composed in a comparable way from derivatives of the equilibrium household quantities with respect to prices and income

$$H \equiv \begin{pmatrix} \theta_p & \theta_P \\ \Theta_p & \Theta_P \end{pmatrix} \qquad h \equiv \begin{pmatrix} \theta_y \\ \Theta_y \end{pmatrix}.$$

This is what would be calculated as the Slutsky matrix if the household were treated as behaving according to the unitary model. The properties of the pseudo-Slutsky matrix can then be examined by relating its terms to the "true" compensated price effects on the functions $f^A(.)$, $f^B(.)$, $F^A(.)$, $F^B(.)$, $G^A(.)$ and $G^B(.)$ which correspond to the individual utility functions assumed to have given rise to the observed behaviour of the household.

Theorem 1 *In an income pooling equilibrium*

1. *(Engel aggregation)* $\qquad \begin{pmatrix} p \\ P \end{pmatrix}' h = 1$

(Cournot aggregation) $\begin{pmatrix} p \\ P \end{pmatrix}' H + \begin{pmatrix} \theta \\ \Theta \end{pmatrix} = 0$

2. *(Homogeneity)* $H \begin{pmatrix} p \\ P \end{pmatrix} + hy = 0$

3. *(Pseudo-Slutsky matrix) Ψ is the sum of a symmetric negative semidefinite matrix and a matrix of rank generically equal to and never more than $n + \min(1, m - \max(n_A - n_B, 1))$.*

Proof of Theorem 1.

See the Appendix. ■

In the general case where the number of private goods is at least two and two partners contribute to roughly similar numbers of public goods then the rank of the departure from a symmetric and negative semidefinite pseudo-Slutsky matrix is therefore $n + 1$, one more than the number of public goods[7].

To reduce the rank of the departure further requires specific restrictions on preferences. Partners' decisions in the equilibrium interact through individually contributed public goods in two ways - firstly, through income effects consequent upon the effect of either individual's spending on household resources effectively available to their partner and, secondly, through effects of individually contributed public good provision on the preference ordering of the partner over remaining goods. We can cut off the latter effect by appropriate separability assumptions and thereby reduce the rank of the departure.

Theorem 2 *In an income pooling equilibrium, if each person's preferences over their contributed goods are separable from the public goods individually contributed by the other or if all public goods are jointly contributed then Ψ is generically the sum of a symmetric negative semidefinite matrix and a matrix of rank no more than 2.*

Proof of Theorem 2.

See the Appendix. ■

This departure from conventional demand properties is a much smaller number but still one more than the same departure in the collective model[8] (Browning and Chiappori [10]). Intermediate cases corresponding to partial separability will give departures of intermediate rank.

[7] Note that in the typical case in which $\max(n_A - n_B, 1) \leq m - 1$ it is immaterial to the rank whether public goods are jointly or individually contributed since, say, an increase in n_X matched by a corresponding fall in $n_A + n_B$ leaves the rank of the departure unchanged.

[8] This is as observed in Lechene and Preston [24], who consider the case in which all public goods are jointly contributed and whose results are substantially generalised by Theorem 1.

4 Separate spheres

4.1 Separate spheres

In separate spheres equilibrium, by contrast, there are no jointly contributed public goods, $n_X = 0$ and partners contribute to disjoint subsets of public goods. The term separate spheres is taken from Lundberg and Pollak [25] who consider such a case as the threat point in a household bargaining model. All goods are individually contributed and household spending on each of the goods to which either partner contributes is effectively constrained by their own individual income. Individual demands follow from solving

$$\max_{q^A, Q_A} u^A(q^A, Q_A, Q_B) \text{ s.t. } \begin{aligned} p'q^A + P'_A Q_A &\leq y^A, \\ Q_A &\geq 0,\ q^A \geq 0 \end{aligned}$$

and

$$\max_{q^B, Q_B} u^B(q^B, Q_A, Q_B) \text{ s.t. } \begin{aligned} p'q^B + P'_B Q_B &\leq y^B, \\ Q_B &\geq 0,\ q^B \geq 0. \end{aligned}$$

and income pooling does not hold. Instead

$$q^A = f^A(y^A, p, P, Q_B) \tag{8}$$
$$Q_A = F^A(y^A, p, P, Q_B) \tag{9}$$
$$q^B = f^B(y^B, p, P, Q_A) \tag{10}$$
$$Q_B = F^B(y^B, p, P, Q_A) \tag{11}$$

and equilibrium demands depend on the distribution of income within the household.

4.2 Household demands

Assuming again uniqueness, we can write the quantities solving the system of equations (8) to (11) as functions $\tilde{\theta}^A(y^A, y^B, p, P)$, $\tilde{\Theta}^A(y^A, y^B, p, P)$, $\tilde{\theta}^B(y^A, y^B, p, P)$ and $\tilde{\Theta}^B(y^A, y^B, p, P)$ and household demands as

$$\begin{aligned} \tilde{\theta}(y^A, y^B, p, P) &= \tilde{\theta}^A(y^A, y^B, p, P) + \tilde{\theta}^B(y^A, y^B, p, P), \\ \tilde{\Theta}(y^A, y^B, p, P) &= \begin{pmatrix} \tilde{\Theta}^A(y^A, y^B, p, P) \\ \tilde{\Theta}^B(y^A, y^B, p, P) \end{pmatrix}. \end{aligned}$$

Even to define conventional derivatives of demand and the pseudo-Slutsky matrix in such a setting presents a problem as these equilibrium demands are not functions of household income.

However household demands can be written as functions of the two individual incomes which together constitute the sole source of variation in household income. It therefore makes sense to look at income responses by considering the rate of change of household demands relative to the rate of change of household income in all directions of variation which hold prices constant - that is to say to consider the set of all directional derivatives [9]

$$\tilde{h}_\phi \equiv \phi^y \begin{pmatrix} \tilde{\theta}_{y^A} \\ \tilde{\Theta}_{y^A} \end{pmatrix} + (1-\phi^y) \begin{pmatrix} \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^B} \end{pmatrix}$$

for some real ϕ^y. Likewise, when considering price responses, we consider the set of directional derivatives in all directions of variation which hold y constant while varying prices:

$$\tilde{H}_\phi \equiv \begin{pmatrix} \tilde{\theta}_p & \tilde{\theta}_P \\ \tilde{\Theta}_p & \tilde{\Theta}_P \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_{y^A} - \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^A} - \tilde{\Theta}_{y^B} \end{pmatrix} \begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix}'$$

for some $\phi^p \in \mathbb{R}^m$ and $\phi^P \in \mathbb{R}^n$. We can then define the set of pseudo-Slutsky matrices

$$\Psi_\phi \equiv \tilde{H}_\phi + \tilde{h}_\phi \begin{pmatrix} \tilde{\theta} \\ \tilde{\Theta} \end{pmatrix}' \tag{12}$$

for all $\phi \equiv \left(\phi^y, \phi^{p\prime}, \phi^{P\prime}\right)' \in \mathbb{R}^{1+m+n}$ and summarise properties of household demand as follows.

Theorem 3 *In separate spheres equilibrium,*

1. *(Engel aggregation)* $\begin{pmatrix} p \\ P \end{pmatrix}' \tilde{h}_\phi = 1$ *for all* $\phi \in \mathbb{R}^{1+m+n}$

   (Cournot aggregation) $\begin{pmatrix} p \\ P \end{pmatrix}' \tilde{H}_\phi + \begin{pmatrix} \theta \\ \Theta \end{pmatrix} = 0$ *for all* $\phi \in \mathbb{R}^{1+m+n}$

2. *(Homogeneity)* $\tilde{H}_\phi \begin{pmatrix} p \\ P \end{pmatrix} + \tilde{h}_\phi y = 0$ *if and only if* $\phi' \begin{pmatrix} y \\ p \\ P \end{pmatrix} = y^A$

3. *(Pseudo-Slutsky matrix)* Ψ_ϕ *is the sum of a symmetric negative semidefinite matrix and a matrix of rank generically equal to and never more than* $n + \min(1, m - \max(n_A - n_B, 1))$ *for all* $\phi \in \mathbb{R}^{1+m+n}$.

Proof of Theorem 3.

[9] Here $\tilde{h}_\phi$ is a directional derivative in the direction $v = \left(\phi^y, 1-\phi^y, 0'_{m+n}\right)'$ in the space of y^A, y^B, p and P, in which a proportion ϕ^y of the change in household income arises from change in y^A. We define the directional derivative in the manner of, say, Apostol [1, p.344], noting that it makes no sense in this context to require $\|v\| = 1$.

See the Appendix. ■

Adding up holds for differentiation in all directions but homogeneity only if the directions are such that equal proportional increases in both household income and prices are associated with increases of the same proportion in the individual incomes[10].

The properties of pseudo-Slutsky matrices, whatever ϕ, depart from those of Slutsky matrices for the unitary household in generically similar fashion to that in the income pooling case. The rank of the departure from conventional demand properties is the same as in the income pooling case - typically $n+1$, one more than the number of public goods. This result therefore applies to any type of equilibrium, and it is therefore immaterial to the generic rank of the departure not only how many public goods are jointly contributed but also whether any are jointly contributed at all. As in the case of income pooling, one would expect this bound on the rank to be generically attained.

Again separability restrictions on preferences will reduce the rank of the departure from unitary properties.

Theorem 4 *In a separate spheres equilibrium, if each person's preferences over their contributed goods are separable from the public goods individually contributed by the other then Ψ_ϕ is generically the sum of a symmetric negative semidefinite matrix and a matrix of rank no more than 1 for all $\phi \in \mathbb{R}^{1+m+n}$.*

Proof of Theorem 4.

See the Appendix. ■

This is now the same rank of departure as found in the collective model[11].

Empirical work conducted under the assumption of unitary decision making on data generated by a separate spheres equilibrium would be misspecified in terms of the household characteristics assumed to determine the outcome. At each level of household income y and of other determinants p, P and so on there will be households choosing differently because they differ in the within-household distribution of income. The properties of estimated Slutsky matrices would depend upon empirical techniques applied. One approach would be to condition on enough additional characteristics[12] in the hope that the division of income within the household could then effectively be treated as a deterministic function of household income and prices. If successful, this approach would give a pseudo-Slutsky matrix of the form Ψ_ϕ with the

[10]Equivalently, the elements of ϕ are equal to the derivatives of a linearly homogeneous function from $\mathbb{R}^{1+m+n}$ to $\mathbb{R}$ taking the value y^A.

[11]The fact that the rank reduction in this case is greater than under income pooling arises because, with no public goods being jointly contributed, this separability restriction is more demanding.

[12]These would be similar to the 'distribution factors' of Browning and Chiappori [10].

elements of ϕ corresponding to partial derivatives of that function with respect to y, p and P. Alternatively, estimation which, say, picked up household demands at the conditional median within-household distribution of income would give rise to pseudo-Slutsky matrices of the form Ψ_ϕ with the elements of ϕ corresponding to median directions of variation in relevant variables. Estimation, on the other hand, which picked out, say, the conditional mean household demand would give a pseudo-Slutsky matrix which would be the mean of matrices of the form Ψ_ϕ and which would therefore typically not have the properties noted in Theorem 3.

5 An example

Consider the following example in illustration of the results derived above. There are m private goods q and one public good Q. Individual preferences are

$$u^J(q^J, Q) \;=\; \alpha^{J\prime} \ln\left(q^J - b^J X\right) + \beta^J \ln Q \qquad J \;=\; A,\; B$$

where $\beta^J = \sum_i \alpha_i^J$ and b^J is an m vector allowing nonseparability of private from public good demand.

Individual demand for someone living alone with income y^J would be

$$\begin{pmatrix} q^J \\ Q \end{pmatrix} = \left[\begin{pmatrix} \xi^J \\ 0 \end{pmatrix} + \begin{pmatrix} b^J \\ 1 \end{pmatrix} \frac{\beta^J}{P + p'b^J}\right] y^J \;\equiv\; \eta^J y^J$$

where $\xi_i^J = \alpha_i^J / p_i$.

In an income pooling equilibrium

$$\begin{pmatrix} q \\ Q \end{pmatrix} = \left[W^A \eta^A \;+\; W^B \eta^B - W^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] y$$

where

$$\begin{aligned} W^0 \;&=\; \beta^A \beta^B / \left[\left(\beta^A + \beta^B - \beta^A \beta^B\right) P \;+\; \beta^B p' b^A \;+\; \beta^A p' b^B\right] \\ W^J \;&=\; W^0 \left(P + p' b^J\right) / \beta^J \qquad J \;=\; A,\; B. \end{aligned}$$

Adding up and homogeneity are easily seen to hold.

The pseudo-Slutsky matrix is

$$\begin{aligned}
\Psi \; &= \; \left[W^A \eta^A_{p,P} \; + \; W^B \eta^B_{p,P} \; + \; W^A_{p,P} \eta^A \; + \; W^B_{p,P} \eta^B - W^0_{p,P} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] y \\
&\quad + \; \left[W^A \eta^A \; + \; W^B \eta^B - W^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] \left[W^A \eta^A \; + \; W^B \eta^B - W^0 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]' y \\
&= \; \Psi^A \; + \; \Psi^B \\
&\quad + \; \beta^A W^A \beta^B W^B \left[\left(\frac{\left(1 - \beta^B\right) R + p b^B}{R + p b^B} \eta^A - \eta^B \right) \left(\eta^A - \begin{pmatrix} b^A \\ 1 \end{pmatrix} \frac{1}{R + p b^A} \right)' \right. \\
&\qquad \left. + \left(\frac{\left(1 - \beta^A\right) R + p b^A}{R + p b^A} \eta^B - \eta^A \right) \left(\eta^B - \begin{pmatrix} b^B \\ 1 \end{pmatrix} \frac{1}{R + p b^B} \right)' \right]
\end{aligned}$$

where

$$\Psi^J \; = \; \left[\eta^J_{p,P} \; + \; \eta^J \eta^{J\prime} \right] W^J y \qquad\qquad J \; = \; A, \; B$$

and

$$\begin{aligned}
\eta^J_{p,P} \; &= \; \begin{pmatrix} -\kappa^J & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} b^J \\ 1 \end{pmatrix} \begin{pmatrix} b^J \\ 1 \end{pmatrix}' \frac{\beta^J}{\left(P + p' b^J\right)^2} && J \; = \; A, \; B \\
\kappa^J_{ij} \; &= \; \xi^J_i / p_i \qquad \text{if} \qquad i = j && \\
&= \; 0 \qquad \text{if} \qquad i \neq j && J \; = \; A, \; B \\
W^J_{p,P} \; &= \; \frac{1}{\beta^J} \left[W^0_{p,P} \left(P + p' b^J\right) \; + \; W^0 \begin{pmatrix} b^J \\ 1 \end{pmatrix}' \right] && J \; = \; A, \; B \\
W^0_{p,P} \; &= \; -\frac{\left(W^0\right)^2}{\beta^A \beta^B} \left[\beta^B \begin{pmatrix} b^A \\ 1 \end{pmatrix} \; + \; \beta^A \begin{pmatrix} b^B \\ 1 \end{pmatrix} \; - \; \beta^A \beta^B \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]'.
\end{aligned}$$

The deviation from symmetry, $\Psi - \Psi^A - \Psi^B$, has rank 2 as suggested by Theorem 1.

In a separate spheres equilibrium with individual A alone contributing to the public good

$$\begin{pmatrix} q^A \\ Q^A \end{pmatrix} \; = \; \left[\begin{pmatrix} \xi^A \\ 0 \end{pmatrix} \; + \; \begin{pmatrix} b^A \\ 1 \end{pmatrix} \frac{\beta^A}{P + p' b^A} \right] y^A \; \equiv \; \eta^A y^A$$

whereas

$$\begin{pmatrix} q^B \\ Q^B \end{pmatrix} \; = \; \begin{pmatrix} \left(I_m - \xi^B p' / \beta^B\right) b^B \\ 0 \end{pmatrix} \frac{\beta^A}{P + p' b^A} y^A \; + \; \begin{pmatrix} \xi^B / \beta^B \\ 0 \end{pmatrix} y^B \; \equiv \; \tilde{\eta}^A y^A \; + \; \tilde{\eta}^B y^B$$

Household demand therefore has the form

$$\begin{pmatrix} q \\ Q \end{pmatrix} = \left[\begin{pmatrix} \xi^A \\ 0 \end{pmatrix} + \begin{pmatrix} b^A + \left(I_m - \xi^B p'/\beta^B\right) b^B \\ 1 \end{pmatrix} \frac{\beta^A}{P + p'b^A} \right] y^A + \begin{pmatrix} \xi^B/\beta^B \\ 0 \end{pmatrix} y^B$$
$$\equiv \left(\eta^A + \tilde{\eta}^A\right) y^A + \tilde{\eta}^B y^B.$$

The pseudo-Slutsky matrix in the direction defined by ϕ is given by

$$\begin{aligned}
\Psi_\phi &= \left[\eta^A_{p,P} + \tilde{\eta}^A_{p,P}\right] y^A + \tilde{\eta}^B_{p,P} y^B + \left[\eta^A + \tilde{\eta}^A - \tilde{\eta}^B\right] \begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix}' \\
&\quad + \left[\phi^y \left(\eta^A + \tilde{\eta}^A\right) + (1 - \phi^y)\, \tilde{\eta}^B\right] \left[\left(\eta^A + \tilde{\eta}^A\right) y^A + \tilde{\eta}^B y^B\right]' \\
&= \Psi^A + \Psi^B \\
&\quad + \tilde{\eta}^A \left[- \begin{pmatrix} b^A \\ 1 \end{pmatrix} + \phi^y \left(\left(\eta^A + \tilde{\eta}^A\right) y^A + \tilde{\eta}^B y^B \right) + \begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix} \right]' \\
&\quad + \left(\eta^A - \tilde{\eta}^B\right) \left[\phi^y \left(\tilde{\eta}^B y^B + \tilde{\eta}^A y^A\right) - (1 - \phi^y)\, \eta^A y^A + \begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix} \right]'
\end{aligned}$$

where

$$\begin{aligned}
\Psi^A &= \left[\eta^A_{p,P} + \eta^A \eta^{A\prime}\right] y^A \\
\Psi^B &= \tilde{\eta}^A_{p,P} y^A + \tilde{\eta}^B_{p,P} y^B + \tilde{\eta}^B \left[\tilde{\eta}^A y^A + \tilde{\eta}^B y^B\right]'
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\eta}^A_{p,P} &= \frac{\beta^A}{P + p'b^A} \left[\begin{pmatrix} \kappa^B p' b^B - \xi^B b^{B\prime} & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\beta^B} - \begin{pmatrix} \left(I_m - \xi^B p'/\beta^B\right) b^B \\ 0 \end{pmatrix} \begin{pmatrix} b^A \\ 1 \end{pmatrix}' \frac{1}{(P + p'b^A)} \right] \\
\tilde{\eta}^B_{p,P} &= \begin{pmatrix} -\kappa^B/\beta^B & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Again the departure from symmetry, $\Psi - \Psi^A - \Psi^B$, is rank 2, in accordance with Theorem 3. If $b^B = 0$ then each person's preferences over those goods which they contribute are separable from the public goods contributed by the other person, $\tilde{\eta}^B = 0$ and the rank of the deviation falls to rank 1 in accordance with Theorem 4.

6 Empirical testing

Theorems 1 and 3 establish a common bound for the rank of the departure of pseudo-Slutsky matrices from symmetric and negative semidefinite matrices in the household Nash equilibrium with voluntary contributed public goods. The bound, which is generically reached, is $n + 1$

(unless $n_A - n_B$ is known and large relative to m). This section assesses the usefulness of this bound for testing either cooperative or noncooperative behaviour.

Note firstly that, unless there are no public goods ($n = 0$), in which case noncooperative behaviour is efficient, then the departure under Nash equilibrium is greater than the rank 1 departure found under the collective model. Browning and Chiappori [10] discuss how to test a rank 1 departure by testing the rank of $\Psi - \Psi'$. If such tests fail to reject a rank 1 departure for couples then the results above establish clearly that cooperative behaviour cannot be rejected against noncooperative behaviour for any number of public goods.

What, however, if cooperative behaviour is rejected? Is it possible to use properties of the pseudo-Slutsky matrix to test compatibility with noncooperative behaviour of the sort analysed here? The first point to note in this respect is that the nature of the departure depends upon the number of public goods. This makes sense. In the cooperative case the rank of the departure is 1 because all interaction arises through the single dimension of the sharing rule. In the noncooperative case interaction arises through the public goods and it is natural that the rank of the departure should depend upon how many public goods there are. There is an important implication of this. Either one knows how many goods are publicly consumed in the household or one can only test noncooperative behaviour jointly with a hypothesis about the number of public goods[13].

The restrictiveness of the above results regarding the properties of pseudo-Slutsky matrices also depends upon not having too many public goods.

Theorem 5 *Let Ψ be either the pseudo-Slutsky matrix in an income pooling equilibrium or a pseudo-Slutsky matrix corresponding to directions of differentiation in which homogeneity holds in a separate spheres equilibrium. Then the restriction that Ψ deviate from symmetry by a matrix of rank no more than $n + 1$ is restrictive if and only if $m \geq n + 5$.*

Proof of Theorem 5.

See the Appendix. ∎

For example, to test for noncooperative behaviour with one public good requires at least 7 goods in total of which 6 are private.

[13] The private or public quality of a good is intrinsically linked with the form of preferences and not the nature of the goods, being a matter of which goods enter the preferences of which individuals. Under the assumption of egoistic preferences, this could be established if it were possible to test the excludability and rivalrousness of those goods.

7 Conclusion

In this paper, we establish properties of demands in the Nash equilibrium with two agents and voluntarily contributed public goods. This noncooperative model is the polar case to the cooperative model of Browning and Chiappori [10] within the class of those models based on individual optimisation.

We show that the nature of the departure from unitary demand properties in household Nash equilibrium is qualitatively similar to that in collectively efficient models in that negativity and symmetry of compensated price responses is not guaranteed. The counterpart to the Slutsky matrix can be shown to depart from negativity and symmetry by a matrix whose rank typically exceeds that found in the collective model unless strong auxiliary restrictions are placed on preferences. This constitutes a testable restriction on household demand functions provided the number of private goods is large enough relative to the number of public goods. Future work will explore sufficient conditions for consistency with noncooperative equilibrium within the household.

Appendix

Proof of Theorem 1.

1. Engel aggregation and Cournot aggregation follow by differentiating the household budget constraint $p'\theta(y,p,P) + P'\Theta(y,p,P) = y$. .

2. Equilibrium quantities satisfying (1) to (6) will satisfy homogeneity given homogeneity of the individual demand functions.

3. Substituting the equilibrium functions into (1) to (6) and differentiating, equilibrium quantity responses are seen to follow from

$$\mathsf{M}\begin{pmatrix} d\theta^A \\ d\Theta^A \\ d\theta^B \\ d\Theta^B \\ d\Xi \end{pmatrix} = \mathsf{N}_1 dy + \mathsf{N}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix} \tag{13}$$

where the matrices M, N_1 and N_2 are defined below.

The M matrix captures interactions between the goods purchases of the two household members and has the form[14]

$$\mathsf{M} \equiv \begin{pmatrix} I & \mathsf{A} & 0 \\ \mathsf{B} & I & 0 \\ 0 & \mathsf{C} & I \\ \mathsf{D} & 0 & I \end{pmatrix}$$

where the non-zero blocks are given by

$$\begin{aligned} \mathsf{A} &= \begin{pmatrix} f_y^A \\ F_y^A \end{pmatrix} \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} 0 & f_{Q_B}^A \\ 0 & F_{Q_B}^A \end{pmatrix} \equiv \mathsf{A}_1 + \mathsf{A}_2, \\ \mathsf{B} &= \begin{pmatrix} f_y^B \\ F_y^B \end{pmatrix} \begin{pmatrix} p \\ P_A \end{pmatrix}' - \begin{pmatrix} 0 & f_{Q_A}^B \\ 0 & F_{Q_A}^B \end{pmatrix} \equiv \mathsf{B}_1 + \mathsf{B}_2, \\ \mathsf{C} &= G_y^A \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} 0 & G_{Q_B}^A \end{pmatrix} \equiv \mathsf{C}_1 + \mathsf{C}_2 \\ \mathsf{D} &= G_y^B \begin{pmatrix} p \\ P_A \end{pmatrix}' - \begin{pmatrix} 0 & G_{Q_A}^B \end{pmatrix} \equiv \mathsf{D}_1 + \mathsf{D}_2. \end{aligned}$$

The components A_1, B_1, C_1 and D_1 arise from interaction through the budget constraint, as greater purchases of any good individually contributed by one partner decreases the amount left over from the household budget for purchases by the other. These matrices each have rank 1, being each the outer product of a vector of income derivatives and a vector of prices.

The components A_2, B_2, C_2 and D_2 arise from the effect of one individual's purchases of individually contributed public goods on the preference ordering of the other over the goods individually contributed by the other. Such terms are generically of rank n_B, $\min(m + n_B, n_A)$, $\min(n_X, n_B)$ and $\min(n_X, n_A)$, respectively[15].

[14] We note the dimension of identity and zero submatrices only where it is not obvious from conformability or from the dimensions of adjacent submatrices.

[15] Each has only n_B, n_A, n_B and n_A non zero columns, respectively, corresponding to the number of public goods individually contributed by the other but in the case of B_2, C_2 and D_2 this determines the rank only if the numbers of rows $m + n_B$, n_X and n_X, respectively, are not short of n_A.

Taking these observations together, we see that the matrices A, B, C and D are therefore generically of rank $1+n_B$, $\min(m+n_B, 1+n_A)$, $\min(n_X, 1+n_B)$ and $\min(n_X, 1+n_A)$,

The N_1 and N_2 matrices take the form

$$\mathsf{N}_1 \equiv \begin{pmatrix} f_y^A \\ F_y^A \\ f_y^B \\ F_y^B \\ G_y^A \\ G_y^B \end{pmatrix} \quad \text{and} \quad \mathsf{N}_2 \equiv \begin{pmatrix} f_p^A - f_y^A q^{B\prime} & f_{P_A}^A & -f_y^A Q_B' & f_R^A \\ F_p^A - F_y^A q^{B\prime} & F_{P_A}^A & -F_y^A Q_B' & F_R^A \\ f_p^B - f_y^B q^{A\prime} & -f_y^B Q_A' & f_{P_B}^B & f_R^B \\ F_p^B - F_y^B q^{A\prime} & -F_y^B Q_A' & F_{P_B}^B & F_R^B \\ G_p^A - G_y^A q^{B\prime} & G_{P_A}^A & -G_y^A Q_B' & G_R^A \\ G_p^B - G_y^B q^{B\prime} & -G_y^B Q_A' & G_{P_B}^B & G_R^A \end{pmatrix}$$

and are composed of conventional income and price effects, excepting that it is necessary to recognise in N_2 that increases in the prices of the public goods individually contributed by one partner decrease the amount left over from the household budget for purchases by the other.

The system (13) is overdetermined, specifically in the final $2n_X$ lines which imply alternative expressions for dX. With $n_X = 1$ compatibility is ensured by adding up, whereas for $n_X > 1$ similar issues arise to those discussed in Section 3.2 concerning the nongenericity of such cases. If we let $\bar{\mathsf{M}}$, $\bar{\mathsf{N}}_1$ and $\bar{\mathsf{N}}_2$ denote the submatrices of M, N_1 and N_2 obtained by deleting the final n_X rows then we can rearrange to get

$$\begin{pmatrix} d\theta^A \\ d\Theta^A \\ d\theta^B \\ d\Theta^B \\ d\Xi \end{pmatrix} = \bar{\mathsf{M}}^{-1}\bar{\mathsf{N}}_1 dy + \bar{\mathsf{M}}^{-1}\bar{\mathsf{N}}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix}.$$

Since we work in terms of household purchases (q, Q_A, Q_B, X), we have therefore

$$\begin{pmatrix} d\theta \\ d\Theta^A \\ d\Theta^B \\ d\Xi \end{pmatrix} = \mathsf{E}\bar{\mathsf{M}}^{-1} \left(\bar{\mathsf{N}}_1 dy + \bar{\mathsf{N}}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix} \right)$$

where

$$\mathsf{E} \equiv \begin{pmatrix} I_m & 0 & I_m & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & 0 & I_{n_B+n_X} \end{pmatrix}$$

is an appropriate aggregating matrix.

The pseudo-Slutsky matrix now follows from:

$$\Psi = \mathsf{E}\bar{\mathsf{M}}^{-1} \left(\bar{\mathsf{N}}_2 + \bar{\mathsf{N}}_1 \begin{pmatrix} q \\ Q_A \\ Q_B \\ X \end{pmatrix}' \right) = \mathsf{E}\bar{\mathsf{M}}^{-1}\Phi$$

where

$$
\begin{aligned}
\Phi &= \begin{pmatrix}
f_p^A + f_y^A q^{A\prime} & f_{P_A}^A + f_y^A Q_A' & 0 & f_R^A + f_y^A X' \\
F_p^A + F_y^A q^{A\prime} & F_{P_A}^A + F_y^A Q_A' & 0 & F_R^A + F_y^A X' \\
f_p^B + f_y^B q^{B\prime} & 0 & f_{P_B}^B + f_y^B Q_B' & f_R^B + f_y^B X' \\
F_p^B + F_y^B q^{B\prime} & 0 & F_{P_B}^B + F_y^B Q_B' & F_R^B + F_y^B X' \\
G_p^A + G_y^A q^{A\prime} & G_{P_A}^A + G_y^A Q_A' & 0 & G_R^A + G_y^A X'
\end{pmatrix} \\
&\equiv \begin{pmatrix}
\Psi_{qq}^A & \Psi_{qQ}^A & 0 & \Psi_{qX}^A \\
\Psi_{Qq}^A & \Psi_{QQ}^A & 0 & \Psi_{QX}^A \\
\Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\
\Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \\
\Psi_{Xq}^A & \Psi_{XQ}^A & 0 & \Psi_{XX}^A
\end{pmatrix}.
\end{aligned}
$$

Note that the terms in Φ are all elements of the underlying symmetric and negative semidefinite true individual conditional Slutsky matrices corresponding to the individual decision problems

$$
\Psi^A \equiv \begin{pmatrix}
\Psi_{qq}^A & \Psi_{qQ}^A & 0 & \Psi_{qX}^A \\
\Psi_{Qq}^A & \Psi_{QQ}^A & 0 & \Psi_{QX}^A \\
0 & 0 & 0_{n_B,n_B} & 0 \\
\Psi_{Xq}^A & \Psi_{XQ}^A & 0 & \Psi_{XX}^A
\end{pmatrix} \text{ and } \Psi^B \equiv \begin{pmatrix}
\Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\
0 & 0_{n_A,n_A} & 0 & 0 \\
\Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \\
\Psi_{Xq}^B & 0 & \Psi_{XQ}^B & \Psi_{XX}^B
\end{pmatrix}.
$$

The matrix $\bar{\mathsf{M}}$ has a block lower triangular structure which helps in inversion. Specifically

$$
\bar{\mathsf{M}}^{-1} = \begin{pmatrix}
I + \mathsf{AB}(I - \mathsf{AB})^{-1} & -\mathsf{A}(I - \mathsf{BA})^{-1} & 0 \\
-\mathsf{B}(I - \mathsf{AB})^{-1} & I + \mathsf{BA}(I - \mathsf{BA})^{-1} & 0 \\
\mathsf{CB}(I - \mathsf{AB})^{-1} & -(\mathsf{C} + \mathsf{CBA}(I - \mathsf{BA})^{-1}) & I
\end{pmatrix}
$$

Thus

$$
\mathsf{E}\bar{\mathsf{M}}^{-1}\Phi = \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \Lambda
$$

where

$$
\begin{aligned}
\Delta_1 &= \mathsf{E}\begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} \mathsf{B}(I - \mathsf{AB})^{-1} \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 & \Psi_{qX}^A \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 & \Psi_{QX}^A \end{pmatrix} \\
\Delta_2 &= \mathsf{E}\begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} \mathsf{A}(I - \mathsf{BA})^{-1} \begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \end{pmatrix} \\
\Lambda &= E\begin{pmatrix} 0_{m+n_A+n_B,m+n} \\ \Lambda_X \end{pmatrix} \\
\Lambda_X &= -\begin{pmatrix} \Psi_{Xq}^B & 0 & \Psi_{XQ}^B & \Psi_{XX}^B \end{pmatrix} \\
&\quad -\left(\mathsf{C} + (\mathsf{CB} + \mathsf{D})\mathsf{A}(I - \mathsf{BA})^{-1} \right) \begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \end{pmatrix}.
\end{aligned}
$$

The individual Slutsky matrices Ψ^A and Ψ^B are symmetric and negative semidefinite and so therefore is their sum. The deviation from conventional demand properties is therefore determined by the properties of $\Delta_1 + \Delta_2 + \Lambda$.

The rank of Δ_1 cannot exceed the rank of B which is at most $\min(1+n_A, m+n_B)$ and that of Δ_2 cannot exceed the rank of A which is at most $1+n_B$, each being defined as products involving these matrices.

The rank of Λ cannot exceed n_X since it contains only n_X non-zero rows. However the rank can be reduced further. Note that, by adding up,

$$\begin{pmatrix} p \\ P_A \\ R \end{pmatrix}' \begin{pmatrix} f_y^A \\ F_y^A \\ G_y^A \end{pmatrix} = \begin{pmatrix} p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} f_y^B \\ F_y^B \\ G_y^B \end{pmatrix} = 1$$

and

$$\begin{pmatrix} p \\ P_A \\ R \end{pmatrix}' \begin{pmatrix} f_{Q_B}^A \\ F_{Q_B}^A \\ G_{Q_B}^A \end{pmatrix} = \begin{pmatrix} p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} f_{Q_A}^B \\ F_{Q_A}^B \\ G_{Q_A}^B \end{pmatrix} = 0.$$

Therefore

$$\begin{aligned} R'\mathsf{C} + \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} &= \left(R'G_y^A + \begin{pmatrix} p \\ P_A \end{pmatrix}' \begin{pmatrix} f_y^A \\ F_y^A \end{pmatrix} \right) \begin{pmatrix} p \\ P_B \end{pmatrix}' \\ &\quad - \left(0 \quad R'G_{Q_B}^A + \begin{pmatrix} p \\ P_A \end{pmatrix}' \begin{pmatrix} f_{Q_B}^A \\ F_{Q_B}^A \end{pmatrix} \right) = \begin{pmatrix} p \\ P_B \end{pmatrix}' \end{aligned}$$

and

$$\begin{aligned} R'\mathsf{D} + \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B} &= \left(R'G_y^B + \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} f_y^B \\ F_y^B \end{pmatrix} \right) \begin{pmatrix} p \\ P_A \end{pmatrix}' \\ &\quad - \left(0 \quad R'G_{Q_A}^B + \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} f_{Q_A}^B \\ F_{Q_A}^B \end{pmatrix} \right) = \begin{pmatrix} p \\ P_A \end{pmatrix}'. \end{aligned}$$

Thus

$$\begin{aligned} R'(\mathsf{C} + (\mathsf{CB}+\mathsf{D})\mathsf{A}(I-\mathsf{BA})^{-1}) &= \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} \\ + \left(\begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B} - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{AB} + \begin{pmatrix} p \\ P_A \end{pmatrix}' - \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B} \right) & \mathsf{A}(I-\mathsf{BA})^{-1} \\ &= \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} + \begin{pmatrix} p \\ P_A \end{pmatrix}' (I-\mathsf{AB})\,(I-\mathsf{AB})^{-1}\mathsf{A} \\ &= \begin{pmatrix} p \\ P_B \end{pmatrix}' \end{aligned}$$

and therefore

$$\begin{aligned} R'\Lambda_X &= -R'(\ \Psi_{Xq}^B \quad 0 \quad \Psi_{XQ}^B \quad \Psi_{XX}^B\ ) - \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \end{pmatrix} \\ &= 0 \end{aligned} \tag{14}$$

by standard properties of the Slutsky matrix. Therefore the rank of Λ_X cannot exceed $n_X - 1$ and neither therefore can that of Λ.

The rank of $\Delta_1 + \Delta_2 + \Lambda$ cannot be greater than the sum of their ranks considered individually which is $n + \min(1, m - n_A + n_B)$. This number cannot exceed the dimension

$n+m$ of the (square) matrix Ψ but can equal it in the one case $m=1$ and $n_A = n_B$. In this case it becomes relevant that Δ_1, Δ_2 and Λ share a common linear dependency since, from (14) and (14)

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E} \begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} = 0$$

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E} \begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} = 0$$

and, from (14),

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E}\Lambda = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \Lambda = R'\Lambda_X = 0.$$

This means that their sum cannot be of full rank and the maximum rank is reduced by 1 in this instance. (This is simply a consequence of Engel and Cournot aggregation which ensure Ψ must be singular as are Ψ^A and Ψ^B.)

The rank of the departure is therefore bounded from above by $n + \min(1, m - \max(n_A - n_B, 1))$.

Although the theorem establishes only a bound on the rank of the sum of the matrices $\Delta_1 + \Delta_2 + \Lambda$, inspection of the form of the matrices suggests that this bound will generically be attained. Generically, only a single public good is jointly contributed in income pooling equilibria in which case Λ disappears. The matrices Δ_1 and Δ_2 are both matrix products in which the factors of lowest rank are B and A respectively and therefore the rank of their sum will generically be the sum of the ranks of A and B.

■

Proof of Theorem 2.

If preferences of each partner over their individually contributed goods are separable from the public goods individually contributed by the other or if there are no individually contributed public goods then A_2 and B_2 disappear, A and B reduce to rank 1 matrices and therefore in the generic case of $n_X = 1$ the rank of the departure falls to 2. ■

Proof of Theorem 3.

1. (Adding up) The household budget constraint requires

$$p'\tilde{\theta}(y^A, y^B, p, P) + P'\tilde{\Theta}(y^A, y^B, p, P) = y^A + y^B$$

and therefore

$$\begin{pmatrix} p \\ P \end{pmatrix}' \begin{pmatrix} \tilde{\theta}_{y^A} \\ \tilde{\Theta}_{y^A} \end{pmatrix} = \begin{pmatrix} p \\ P \end{pmatrix}' \begin{pmatrix} \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^B} \end{pmatrix} = 1$$

from which Engel and Cournot aggregation follow for any ϕ.

2. (Homogeneity) Since individual demand functions are homogeneous of degree zero

$$\tilde{\theta}(\lambda y^A, \lambda y^B, \lambda p, \lambda P) = \theta(y^A, y^B, p, P) \quad \text{and} \quad \tilde{\Theta}(\lambda y^A, \lambda y^B, \lambda p, \lambda P) = \Theta(y^A, y^B, p, P)$$

for any $\lambda > 0$. Therefore

$$\begin{pmatrix} \tilde{\theta}_p & \tilde{\theta}_P \\ \tilde{\Theta}_p & \tilde{\Theta}_P \end{pmatrix} \begin{pmatrix} p \\ P \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_{y^A} \\ \tilde{\Theta}_{y^A} \end{pmatrix} y^A + \begin{pmatrix} \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^B} \end{pmatrix} y^B = 0$$

and $\tilde{H}_\phi \begin{pmatrix} p \\ P \end{pmatrix} + \tilde{h}_\phi y = 0$ only if

$$\begin{aligned} \left(p'\phi^p + P'\phi^P\right) \begin{pmatrix} \tilde{\theta}_{y^A} - \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^A} - \tilde{\Theta}_{y^B} \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_{y^A} \\ \tilde{\Theta}_{y^A} \end{pmatrix} \phi^y y + \begin{pmatrix} \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^B} \end{pmatrix} (1 - \phi^y) y \\ = \begin{pmatrix} \tilde{\theta}_{y^A} \\ \tilde{\Theta}_{y^A} \end{pmatrix} y^A + \begin{pmatrix} \tilde{\theta}_{y^B} \\ \tilde{\Theta}_{y^B} \end{pmatrix} y^B \end{aligned}$$

which is true only if $\phi^y y + p'\phi^p + P'\phi^P = y^A$.

3. As in the income pooling case, we substitute the equilibrium functions into (8) to (11) and differentiate, to derive

$$\mathcal{M} \begin{pmatrix} d\tilde{\theta}^A \\ d\tilde{\Theta}^A \\ d\tilde{\theta}^B \\ d\tilde{\Theta}^B \end{pmatrix} = \mathcal{N}_1^A dy^A + \mathcal{N}_1^B dy^B + \mathcal{N}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \end{pmatrix} \tag{15}$$

where the matrices $\mathcal{M}$, $\mathcal{N}_1^A$, $\mathcal{N}_1^B$ and $\mathcal{N}_2$ are defined below.

The $\mathcal{M}$ matrix captures interactions between the goods purchases of the two household members and has the form

$$\mathcal{M} \equiv \begin{pmatrix} I & \mathcal{A} \\ \mathcal{B} & I \end{pmatrix}$$

where the off-diagonal blocks are given by

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 0 & f^A_{Q_B} \\ 0 & F^A_{Q_B} \end{pmatrix}, \\ \mathcal{B} &= \begin{pmatrix} 0 & f^B_{Q_A} \\ 0 & F^B_{Q_A} \end{pmatrix} \end{aligned}$$

and capture only the effects of individually contributed public goods purchases on partners' preference orderings over their individually contributed goods. $\mathcal{A}$ and $\mathcal{B}$ are generically of rank n_B and $\min(n_A, m + n_B)$, respectively.

The $\mathcal{N}_1^A$, $\mathcal{N}_1^B$ and $\mathcal{N}_2$ matrices take the forms

$$\mathcal{N}_1^A \equiv \begin{pmatrix} f^A_y \\ F^A_y \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{N}_1^B \equiv \begin{pmatrix} 0 \\ 0 \\ f^B_y \\ F^B_y \end{pmatrix}, \quad \mathcal{N}_2 \equiv \begin{pmatrix} f^A_p & f^A_{P_A} & 0 \\ F^A_p & F^A_{P_A} & 0 \\ f^B_p & 0 & f^B_{P_B} \\ F^B_p & 0 & F^B_{P_B} \end{pmatrix}$$

and are composed of conventional income and price effects.
Defining a suitable aggregating matrix

$$\mathcal{E} \equiv \begin{pmatrix} I_m & 0 & I_m & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & 0 & I_{n_B} \end{pmatrix}$$

we can derive the pseudo-Slutsky matrices for the separate spheres case:

$$\begin{aligned}
\Psi_\phi &= \mathcal{E}\mathcal{M}^{-1}\Bigg(\mathcal{N}_2 + \left[\mathcal{N}_1^A - \mathcal{N}_1^B\right]\begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix}' \\
&\qquad + \left[\phi^y\mathcal{N}_1^A + (1-\phi^y)\mathcal{N}_1^B\right]\begin{pmatrix} q \\ Q_A \\ Q_B \end{pmatrix}'\Bigg) \\
&= \mathcal{E}\mathcal{M}^{-1}\left[\Phi + (\mathcal{N}_1^A - \mathcal{N}_1^B)\zeta'\right]
\end{aligned}$$

where

$$\Phi = \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 \\ \Psi_{qq}^B & 0 & \Psi_{qQ}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B \end{pmatrix}$$

and

$$\zeta_A = \begin{pmatrix} q^B \\ 0 \\ Q^B \end{pmatrix}, \quad \zeta_B = \begin{pmatrix} q^A \\ Q^A \\ 0 \end{pmatrix}, \quad \zeta = \phi^y\zeta^A - (1-\phi^y)\zeta^B + \begin{pmatrix} \phi_p \\ \phi_P \end{pmatrix}.$$

As in the income pooling case, Φ is made up of components of the true conditional Slutsky matrices

$$\Psi^A \equiv \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 \\ 0 & 0 & 0_{n_B,n_B} \end{pmatrix} \text{ and } \Psi^B \equiv \begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B \\ 0 & 0_{n_A,n_A} & 0 \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B \end{pmatrix}.$$

while ζ_A, ζ_B and ζ are vectors.
The inverse of $\mathcal{M}$ has the form

$$\mathcal{M}^{-1} = \begin{pmatrix} I + \mathcal{A}\mathcal{B}(I-\mathcal{A}\mathcal{B})^{-1} & -\mathcal{A}(I-\mathcal{B}\mathcal{A})^{-1} \\ -\mathcal{B}(I-\mathcal{A}\mathcal{B})^{-1} & I + \mathcal{B}\mathcal{A}(I-\mathcal{B}\mathcal{A})^{-1} \end{pmatrix}$$

Thus

$$\Psi_\phi = \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \mathcal{K}$$

where

$$\begin{aligned}
\Delta_1 &= \mathcal{E}\begin{pmatrix} \mathcal{A} \\ -I \end{pmatrix}\mathcal{B}(I-\mathcal{A}\mathcal{B})^{-1}\left[\begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 \end{pmatrix} + \begin{pmatrix} f_y^A \\ F_y^A \end{pmatrix}\zeta'\right] \\
\Delta_2 &= \mathcal{E}\begin{pmatrix} -I \\ \mathcal{B} \end{pmatrix}\mathcal{A}(I-\mathcal{B}\mathcal{A})^{-1}\left[\begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B \end{pmatrix} - \begin{pmatrix} f_y^B \\ F_y^B \end{pmatrix}\zeta'\right]
\end{aligned}$$

and

$$\mathcal{K} \;=\; \mathcal{E}\left[\mathcal{N}_1^A - \mathcal{N}_1^B\right]\zeta'.$$

The individual Slutsky matrices Ψ^A and Ψ^B are symmetric and negative semidefinite and so therefore is their sum. The deviation from conventional demand properties is therefore determined by the properties of $\Delta_1 + \Delta_2 + \mathcal{K}$.

The rank of Δ_1 cannot exceed the rank of $\mathcal{B}$ which is at most $\min(n_A, m + n_B)$ and that of Δ_2 cannot exceed the rank of $\mathcal{A}$ which is at most n_B, each being defined as products involving these matrices. Moreover, $\mathcal{K}$ being a matrix product involving an outer product of vectors[16], has rank 1.

The rank of $\Delta_1 + \Delta_2 + \mathcal{K}$ cannot be greater than the sum of the ranks of the component matrices considered individually which is $n + \min(1, m - n_A + n_B)$. This number cannot exceed the dimension $n + m$ of the (square) matrix Ψ but can equal it in the one case $m = 1$ and $n_A = n_B$. In this case it becomes relevant that Δ_1, Δ_2 and $\mathcal{K}$ all share a common linear dependency since, by adding up,

$$\left(\begin{array}{c} p \\ P_A \end{array}\right)'\left(\begin{array}{c} f^A_{Q_B} \\ F^A_{Q_B} \end{array}\right) = \left(\begin{array}{c} p \\ P_B \end{array}\right)'\left(\begin{array}{c} f^B_{Q_A} \\ F^B_{Q_A} \end{array}\right) = 0$$

and therefore

$$\begin{aligned}
\left(\begin{array}{c} p \\ P \end{array}\right)'\mathcal{E}\left(\begin{array}{c} \mathcal{A} \\ -I \end{array}\right)\mathcal{B} = \left(\begin{array}{c} p \\ P_A \\ p \\ P_B \end{array}\right)'\left(\begin{array}{c} \mathcal{A} \\ -I \end{array}\right)\mathcal{B} &= -\left(\begin{array}{c} p \\ P_B \end{array}\right)'\mathcal{B} = 0 \\
\left(\begin{array}{c} p \\ P \end{array}\right)'\mathcal{E}\left(\begin{array}{c} -I \\ \mathcal{B} \end{array}\right)\mathcal{A} = \left(\begin{array}{c} p \\ P_A \\ p \\ P_B \end{array}\right)'\left(\begin{array}{c} -I \\ \mathcal{B} \end{array}\right)\mathcal{A} &= -\left(\begin{array}{c} p \\ P_A \end{array}\right)'\mathcal{A} = 0
\end{aligned}$$

and

$$\left(\begin{array}{c} p \\ P \end{array}\right)'\mathcal{E}\left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] = \left(\begin{array}{c} p \\ P_A \\ p \\ P_B \end{array}\right)'\left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] = 1 - 1 = 0.$$

Thus the maximum rank is reduced by 1 in this instance and the rank of the departure is therefore bounded from above by $n + \min(1, m - \max(n_A - n_B, 1))$.

The rank of $\mathcal{K}$ is always 1. The matrices Δ_1 and Δ_2, as in the income pooling case, are both matrix products and their sum will generically have rank equal to the sum of the ranks of $\mathcal{A}$ and $\mathcal{B}$.

■

Proof of Theorem 4.

If preferences of each partner over their individually contributed goods are separable from the public goods individually contributed by the other then $\mathcal{A}$ and $\mathcal{B}$ disappear and therefore the rank of the departure falls to 1. ■

Lemma 1 *Let Ψ be a real $k \times k$ matrix such that the rank of $\Psi - \Psi'$ cannot exceed s. Then Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r if and only if either (i) $2r + 1 \geq s$ or (ii) $2r + 1 < s$ and $\Psi - \Psi'$ has rank at most $2r$.*

[16] By adding up, it is impossible for either $\mathcal{N}_1^A$ or $\mathcal{N}_1^B$ to be zero vectors or for them to equal each other so these matrices have rank of exactly 1.

Proof of Lemma 1.

For any real $k \times k$ matrix Ψ the matrix $\Psi - \Psi'$ is skew symmetric and therefore has even rank. If the rank of $\Psi - \Psi'$ cannot exceed s then its rank is therefore at most s if s is even and at most $s - 1$ if s is odd.

Suppose Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r. Then

$$\Psi = S + \sum_{i=1}^{r} u_i v_i'$$

where $S = S'$ and u_i, v_i are $k \times 1$ vectors, $i = 1, \ldots, r$. Then

$$\Psi - \Psi' = \sum_{i=1}^{r} (u_i v_i' - v_i u_i')$$

which has rank at most $2r$. If s is even then the rank is therefore at most $\min(2r, s)$ whereas if s is odd then the rank is therefore at most $\min(2r, s-1)$. In each case the bound of $2r$ is restrictive only if $2r + 1 < s$.

Conversely, suppose $\Psi - \Psi'$ has rank at most $2r$. (Note that this holds for any matrix Ψ if the rank of $\Psi - \Psi'$ cannot exceed s and $2r + 1 \geq s$.) Since $\Psi - \Psi'$ is real and skew symmetric, it is possible (see, for example, Theorem 2.5 in Thompson [31]) to write $\Psi - \Psi' = ULU'$ for some orthogonal matrix U and a block diagonal matrix $L = \mathrm{diag}(L_1, \ldots, L_r, 0, \ldots, 0)$ where

$$L_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

for some real λ_i, $i = 1, \ldots, r$. Therefore $\Psi - \Psi' = \sum_{i=1}^{r} \lambda_i (u_i v_i' - v_i u_i')$ where $U = (u_1\ v_1\ u_2\ v_2\ \ldots)$. Then $\Psi - \sum_{i=1}^{r} \lambda_i u_i v_i' = \Psi' - \sum_{i=1}^{r} \lambda_i v_i u_i'$ is symmetric. Call this matrix S. Then Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r

$$\Psi = S + \sum_{i=1}^{r} \lambda_i u_i v_i'.$$

This result generalises Lemma 1 of Browning and Chiappori [10] to cover departures of *any* rank. ■

Proof of Theorem 5.

If demands satisfy adding up then $\begin{pmatrix} p \\ P \end{pmatrix}' \Psi = 0$ and if they satisfy homogeneity then $\Psi \begin{pmatrix} p \\ P \end{pmatrix} = 0$. Therefore $\begin{pmatrix} p \\ P \end{pmatrix}' (\Psi - \Psi') = 0$ and the rank of $\Psi - \Psi'$ cannot exceed one less than its dimension $n + m$.

Lemma 1 shows that if $\Psi - \Psi'$ is known to have rank at most $n + m - 1$ then it is possible to write Ψ as the sum of a symmetric matrix and a matrix of rank at most r whenever $2r + 1 \geq n + m - 1$. A departure from symmetry of rank at most $n + 1$ is therefore restrictive only if $2n + 3 < n + m - 1$. It must therefore be the case that $m \geq n + 5$ in order to test for noncooperative behaviour with n public goods. ■

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