DEMAND PROPERTIES IN HOUSEHOLD NASH EQUILIBRIUM

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Demand Properties in Household Nash Equilibrium *

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Abstract

We study noncooperative household models with two agents and several voluntarily contributed public goods, deriving the counterpart to the Slutsky matrix and demonstrating the nature of the deviation of its properties from those of a true Slutsky matrix in the unitary model. We provide results characterising both cases in which there are and are not jointly contributed public goods. Demand properties are contrasted with those for collective models and conclusions drawn regarding the possibility of empirically testing the collective model against noncooperative alternatives.

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1 Introduction

Maximisation of utility by a single consumer subject to a linear budget constraint implies strong testable restrictions on the properties of demand functions. Empirical applications to data on households however often reject these restrictions. In particular, such data frequently show a failure of Slutsky symmetry - the restriction of symmetry on the matrix of compensated price responses (see for example Deaton (1990), Browning and Meghir (1991), Banks, Blundell and Lewbel (1997) and Browning and Chiappori (1998)).

From the theoretical point of view, the inadequacy of the single consumer model as a description of decision making for households with more than one member has also long been recognised. Attempts to reconcile this model with the existence of several sets of individual preferences have been made for instance by Samuelson (1956) and Becker (1974, 1991) but rely upon restrictive assumptions about preferences or within-household decision mechanisms (see Bergstrom, 1989; Cornes and Silva, 1999).

A large body of recent research has investigated models accommodating alternative descriptions of within-household decision-making processes. Efficiency of household decisions holds in a number of models of household behaviour which have been suggested: for instance in the Nash bargaining models of Manser and Brown (1980), McElroy and Horney (1981) and McElroy (1990), and in Browning, Bourguignon, Chiappori and Lechene (1994) and Bourguignon and Chiappori (1994). However, it is not a property of noncooperative models such as those of Ulph (1988) and Chen and Woolley (2001).

An important advance is made by Browning and Chiappori (1998), who show that under the sole assumption of efficient within-household decision making, the counterpart to the Slutsky matrix for demands from a k member household is the sum of a symmetric matrix and a matrix of rank $k-1$. Chiappori and Ekeland (2006a) establish not only that efficiency implies a rank $k-1$ deviation but also that a rank $k-1$ deviation implies the existence of preferences compatible with efficient behaviour. Chiappori and Ekeland (2006b) show that for these preferences to be identified it is required to know which goods are private and which are public and that it is sufficient for identification to assume the existence of exclusive goods. Browning and Chiappori report tests on Canadian data which reject symmetry for couples, but not for single individual households. The hypothesis that the departure from symmetry for the sample of couples has rank 1, as implied by the assumption of efficiency, is also not rejected.

These results not only fill a gap in our theoretical understanding of demand behaviour but also open the prospect of reconciling demand theory and data on consumer behaviour. The work of Browning and Chiappori (1998) and Chiappori and Ekeland (2006a) is important in showing that the assumption of efficiency generates testable restrictions on household demand functions, clearly distinguishing the collective model from both the unitary and the entirely unrestricted case.

In this paper we explore the same question of the testable restrictions implied by an alternative structural assumption on within household interactions. The model considered is the principal alternative to both the unitary and collective models, that of noncooperative demand behaviour with voluntarily contributed public goods. This model warrants attention in its own right as the only currently widely discussed alternative to fully efficient models of the sort described above. It is also interesting in so far as the equilibria in this model can be considered as the fallback position in bargaining models as suggested, for example, in Woolley (1988), Lundberg and Pollak (1993) and Chen and Woolley (2001).

Models of voluntarily contributed public goods have relevance beyond analysis of household demand. When they involve more than two players, these models can be used to represent a variety of situations involving private contributions to public goods either in the national or international context. What distinguishes what we have termed the "household Nash equilibrium model" from the general Nash equilibrium model is the small number of agents, which is two in the case considered here.

We derive results for all types of equilibria, including those in which partners do and do not contribute jointly to a common set of public goods. In section 2 we lay out the general framework. In section 3, we consider the case in which there are jointly contributed public goods. We show that equilibrium quantities vary with prices and household income in ways compatible with the adding up and homogeneity properties of unitary demands and that negativity and symmetry properties will generally be violated, as in the collective model. We derive the counterpart to the Slutsky matrix, and show that it can be decomposed into the sum of a symmetric matrix and another of bounded rank, though its rank generally exceeds the deviation to be expected in a collective setting. We establish also the numbers of public and private goods required for this to constitute a testable restriction on behaviour. Section 4 is devoted to the properties of demands in the case of no jointly contributed public goods. Adding up holds, homogeneity may fail and the rank of the departure from negativity and symmetry is shown to be similar in rank to that when public goods are jointly contributed. These results suggest that the properties of the Slutsky matrix do provide a potential basis for testing not only the Browning-Chiappori assumption of efficiency but also other models within the class of those based on individual optimisation. Section 6 concludes.

2 The general model

Consider a household with two individuals, A and B. The household spends on a set of m private goods $q \in \mathbb{R}^m$ and n public goods $Q \in \mathbb{R}^n$. The quantities of private goods purchased by the individuals are q^A and q^B with total household quantities $q \equiv q^A + q^B$. The quantities of public goods purchased by the household are Q with individual contributions Q^A and Q^B and $Q \equiv Q^A + Q^B$. Individual utility functions are $u^A(q^A, Q)$ and $u^B(q^B, Q)$, assumed increasing and differentiable in all arguments, so that individual preferences are defined over the sum of contributions to the public goods. The partners have incomes of y^A and y^B. Household income is denoted $y \equiv y^A + y^B$. Prices of private and public goods respectively are the vectors p and P.

Each person decides on the purchases made from their income so as to maximise their utility subject to the spending decisions of their partner. We can write the agents' problems as

$$\max_{q^A, Q^A} u^A(q^A, Q) \text{ s. t. } p'q^A + P'Q^A \leq y^A, \; Q^A \geq 0, \; q^A \geq 0$$

and

$$\max_{q^B, Q^B} u^B(q^B, Q) \text{ s. t. } p'q^B + P'Q^B \leq y^B, \; Q^B \geq 0, \; q^B \geq 0$$

where inequalities should be read where appropriate as applying to each element of the relevant vector.

A *household Nash equilibrium* consists of a set of quantities (q^A, q^B, Q^A, Q^B) simultaneously solving these two problems. The existence of at least one such equilibrium is established in Browning, Chiappori and Lechene (2005). The equilibrium need not be unique, though Bergstrom, Blume and Varian (1986, 1992), Fraser (1992) and Lechene and Preston (2005) provide sufficient conditions, essentially involving normality of both public and private goods, for uniqueness of certain sorts of equilibria.

In any equilibrium, public goods can be divided into two types - those to which only one partner contributes and those to which both do. We refer to the former as *individually contributed public goods*, and denote the quantity vectors for such goods[1] Q_A and Q_B, the respective prices P_A and P_B and their dimensions n_A and n_B. Without loss of generality we assume $n_A \geq n_B$. The latter type, on the other hand, are referred to as *jointly contributed public goods*, with quantity vector denoted X, prices R and dimension n_X. Individual contributions to these public goods are denoted X^A and X^B.

Equilibria can be distinguished into those in which there are and are not jointly contributed public goods. Those in which $n_X \geq 1$ are called, for reasons which will become apparent, *income pooling equilibria* and those in which $n_X = 0$ are called *separate spheres equilibria*.

It is useful to recognise that the problems can be rewritten to have partners effectively choosing the levels of the public goods for the household, subject to the constraint that this level is greater than or equal to the contribution of the other agent. Given that $y^A = y - p'q^B - P'Q^B$, and similarly for B, the agents'problems can be re-written as:

$$\max_{q^A, Q} u^A(q^A, Q) \text{ s. t. } p'q^A + P'Q \leq y - p'q^B,\ Q \geq Q^B,\ q^A \geq 0$$

and

$$\max_{q^B, Q} u^B(q^B, Q) \text{ s. t. } p'q^B + P'Q \leq y - p'q^A,\ Q \geq Q^A\ q^B \geq 0.$$

3 Income pooling equilibria

3.1 Income pooling

In an income pooling equilibrium, the solution to each partner's problem coincides with that in which they choose only over their privately contributed

[1] Note that subscripts A and B are used to distinguish goods contributed exclusively by individuals A and B whereas superscripts A and B distinguish contributions by individuals A and B (to any good).

goods and the jointly contributed public good X

$$\max_{q^A, Q_A, X} u^A(q^A, Q_A, Q_B, X) \text{ s. t. } p'q^A + P'_A Q_A + R'X \leq y - p'q^B - P'_B Q_B, \quad Q_A \geq 0,\ X \geq X^B,\ q^A \geq 0$$

and

$$\max_{q^B, Q_B, X} u^B(q^B, Q_A, Q_B, X) \text{ s. t. } p'q^B + P'_B Q_B + R'X \leq y - p'q^A - P'_A Q_A, \quad Q_B \geq 0,\ X \geq X^A,\ q^B \geq 0.$$

Hence, quantities purchased will satisfy

$$\begin{aligned} q^A &= f^A(y - p'q^B - P'_B Q_B, p, P, Q_B) && (1) \\ Q_A &= F^A(y - p'q^B - P'_B Q_B, p, P, Q_B) && (2) \\ q^B &= f^B(y - p'q^A - P'_A Q_A, p, P, Q_A) && (3) \\ Q_B &= F^B(y - p'q^A - P'_A Q_A, p, P, Q_A) && (4) \end{aligned}$$

and

$$\begin{aligned} X &= G^A(y - p'q^B - P'_B Q_B, p, P, Q_B) && (5) \\ &= G^B(y - p'q^A - P'_A Q_A, p, P, Q_A). && (6) \end{aligned}$$

where $f^A(.)$, $F^A(.)$, $f^B(.)$, $F^B(.)$, $G^A(.)$, $G^A(.)$ are conditional Marshallian demand functions corresponding to the two partners' preferences and together satisfying the usual demand properties.

We use subscripts to denote derivatives of these demand functions: f^i_y, f^i_p, f^i_P, $f^i_{Q_j}$, F^i_y, F^i_p, F^i_P, $F^i_{Q_j}$ and G^i_y, G^i_p, G^i_P, $G^i_{Q_j}$ for $i = A, B$, with respect to income y, price vectors p and P and individually contributed public goods quantities of the other partner Q_j respectively.

Note that (1) to (4) define $2m+n_A+n_B$ equilibrium equations in $2m+n_A+n_B$ quantities (q^A, Q_A, q^B, Q_B) independently of (5) and (6). Substituting solutions to these equations into (5) or (6) will give the set of income pooling equilibria. Furthermore the set of solutions to these equations plainly depend only upon (y, p, P) and in particular do not depend upon the distribution of income within the household. This well known "income pooling" result is the source of the name given to such equilibria. This result is well known and has been discussed by many authors. Warr (1983) established income pooling for the case of a single public good and Kemp (1984) extended the claim to the case of multiple public goods, assuming interior equilibrium. Kemp's proof is queried by Bergstrom, Blume and Varian (1986) who offer an alternative proof.

Though often found surprising, the source of the result is easily illustrated[2] for the case of only one private good and one jointly contributed public good in Figure 1. Any allocation of total household income y across the three goods q^A, q^B and X can be represented as a point in the triangular area ADO with the shares of household income spent on private goods represented by the distances along the axes and the remaining share allocated to the public good given by the perpendicular distance to the boundary AD. Given any amount spent on the private good of individual A, the remainder of household income is spent on goods of interest to individual B and the line AEB represents B's preferred allocation between q^B and X. Correspondingly, the line DEC represents A's preferred allocation between q^A and X given any amount spent on q^B. The line AEB and DEC represent graphically the reaction functions implied by equations (1) and (2). The intersection at E shows an allocation over the three goods with which each partner is content given the spending decisions of the other. This point is clearly unique if the slope of AEB is always more negative than -1 and that of DEC always between 0 and -1 which will be the case if X and q are normal in the preferences of A and B. This point will be an income pooling equilibrium if it involves neither partner spending more than their private income on their own private good. Individual income shares y^A/y and y^B/y are shown on the diagram and in this case exceed household budget shares for the private goods at E so that E is the unique household Nash equilibrium. Furthermore it is clear that small changes in the income shares will not alter the location of this equilibrium, which is the income pooling result. Separate spheres equilibria will pertain in cases of sufficiently extreme income shares and the locus of all equilibria is given by the line CEB.

[2] Ley (1996) presents several diagrammatic representations of the Bergstrom, Blume and Varian (1986) model though not that of Figure 1.

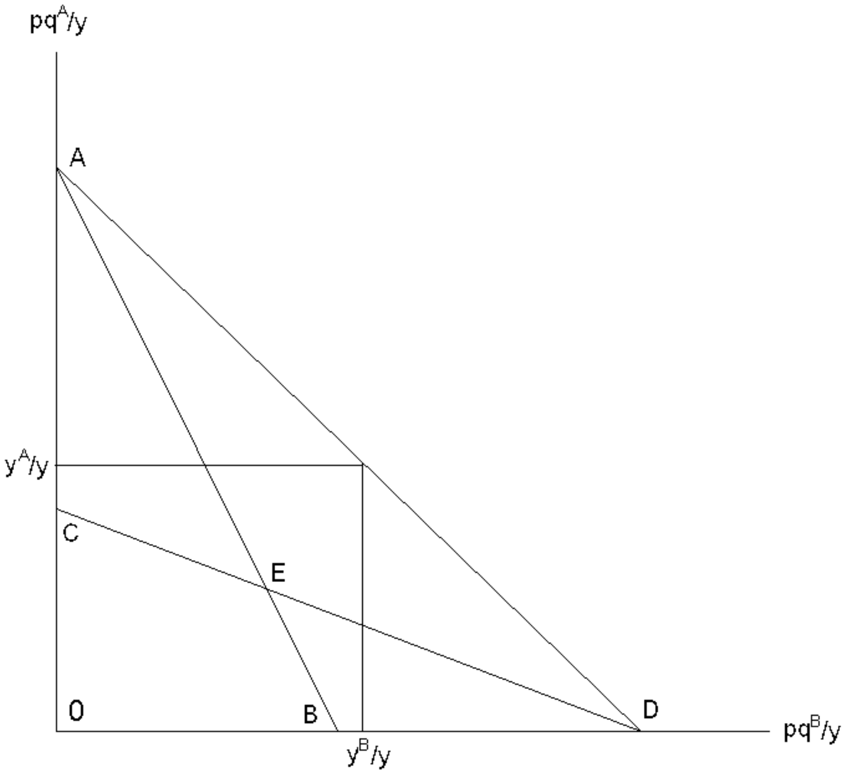

Figure 1: Household Nash equilibrium

3.2 Jointly contributed public goods

Satisfaction of both (5) and(6) with multiple jointly contributed public goods at anything other than isolated values of (y, p, P) clearly requires a certain coincidence in preferences over public goods between the two partners. Browning, Chiappori and Lechene (2005) demonstrate that generically $n_X \leq 1$ so that typically there will not exist more than a single jointly contributed public good in equilibrium. More precisely, given a suitable topology on preferences, there is no open set in the space of the couple's preferences, incomes and prices on which $n_X > 1$ in equilibrium. This is not to say, however, that there are not subspaces of preferences within which equilibria with $n_X > 1$ can hold on an open set of values for (y, p, P). What is required is that the partners' marginal rates of substitution between jointly contributed public goods should coincide at all equilibrium quantities of the goods over such a set. That is possible, for example, if preferences over jointly contributed public goods are separable and identical for the two partners. It is, in fact, possible even without such separability if preferences over those individually contributed goods from which there is not separability are also identical between the partners since there exist equilibria with quantities of these goods also identical in equilibrium[3]. Lechene and Preston (2006) demonstrate the possibility of such cases. Of course, these cases are not robust to small independent perturbations in the partners' preferences but identity and separability of preferences over subsets of public goods may make sense in certain cases - for example, if the subutility function reflects an agreed technology for producing some intermediate good or if, say, the goods in question relate to children and the subutility reflects an agreed welfare function for the children. In any case, we present results covering both the generic case and the possibility that $n_X > 1$.

[3]Specifically, interior equilibria with $n_X > 1$ exist on an open set of values for (y, p, P) if private goods can be partitioned, $q^i = (q_0^i, q_1^i)$, $i = A, B$, in such a way that individual preferences take the weakly separable form

$$u^i(q^i, Q) = \upsilon^i(q_0^i, Q^i, \nu(q_1^i, X)) \qquad i = A, B$$

for some $\upsilon^i(.,.,.)$, $i = A, B$ and some common subutility function $\nu(.,.)$. In such a case, marginal rates of substitution between public goods in X are, for each partner, the same function of quantities q_1^i and X and there exist equilibria with $q_1^A = q_1^B$ so that these marginal rates of substitution coincide as required. Such preferences obviously include, for instance, the cases both of common separability of public goods ($q^i = q_0^i$, $X = Q$) and of identical preferences ($q^i = q_1^i$, $X = Q$).

3.3 Adding up and homogeneity

In what follows we assume uniqueness of the equilibrium and denote the mappings from (y, p, P) to the unique individual equilibrium goods vectors by $\theta^A(y, p, P)$, $\Theta^A(y, p, P)$, $\theta^B(y, p, P)$, $\Theta^B(y, p, P)$ and to the jointly contributed quantities by $\Xi(y, p, P)$. We let

$$\theta(y, p, P) = \theta^A(y, p, P) + \theta^B(y, p, P)$$

and

$$\Theta(y, p, P) = \begin{pmatrix} \Theta^A(y, p, P) \\ \Theta^B(y, p, P) \\ \Xi(y, p, P) \end{pmatrix}$$

denote the household private and public goods vectors. Note that quantities are uniquely determined as functions of the same economic determinants y, p and P as would be the case under the "unitary" model where the household maximises a household utility function given the household budget constraint. Distinguishing unitary and noncooperative household behaviour therefore requires that we establish whether these equilibrium quantities have properties dissimilar to demands in unitary households. Browning and Chiappori (1998) provide such an analysis for the collective model, and Browning, Chiappori and Lechene (2004) examine the relationship between collective and unitary models.

The properties of unitary demands are the standard Hurwicz-Uzawa (1971) integrability requirements of adding up, homogeneity, negativity and symmetry. It is easy to establish that the household Nash equilibrium quantities satisfy adding-up and homogeneity.

Theorem 1 *In income pooling equilibrium, household Nash equilibrium demands satisfy*

1. *(Adding up)* $p'\theta(y, p, P) + P'\Theta(y, p, P) = y$
2. *(Homogeneity)* $\theta(\lambda y, \lambda p, \lambda P) = \theta(y, p, P)$ *and* $\Theta(\lambda y, \lambda p, \lambda P) = \Theta(y, p, P)$ *for any* $\lambda > 0$.

Proof of Theorem 1.

1. Adding up of demands in household Nash equilibrium follows from the fact that the partners are on their individual budget constraints and the sum of their demands therefore satisfies the household budget constraint.

2. Equilibrium quantities satisfying (1) to (6) will satisfy homogeneity given homogeneity of the individual demand functions.

■

3.4 Negativity and symmetry

Negativity and symmetry are less simply dealt with. These are concerned in the case of the unitary model with the properties of the Slutsky matrix, the matrix of price responses at fixed household utility. Since household utility is undefined in a noncooperative setting, no such matrix is defined but we can adopt the Browning and Chiappori (1998) notion of the "pseudo-Slutsky matrix". This in the current context is the matrix

$$\Psi \equiv \begin{pmatrix} \theta_p & \theta_P \\ \Theta_p & \Theta_P \end{pmatrix} + \begin{pmatrix} \theta_y \\ \Theta_y \end{pmatrix} \begin{pmatrix} \theta \\ \Theta \end{pmatrix}' \tag{7}$$

composed in a comparable way from derivatives of the equilibrium household quantities with respect to prices and income. This is what would be calculated as the Slutsky matrix if the household were treated as behaving according to the unitary model. The properties of the pseudo-Slutsky matrix can then be examined by relating its terms to the "true" compensated price effects on the functions $f^A(.)$, $f^B(.)$, $F^A(.)$, $F^B(.)$, $G^A(.)$ and $G^B(.)$ which correspond to the individual utility functions assumed to have given rise to the observed behaviour of the household.

Substituting the equilibrium functions into (1) to (6) and differentiating, equilibrium quantity responses are seen to follow from

$$\mathsf{M} \begin{pmatrix} d\theta^A \\ d\Theta^A \\ d\theta^B \\ d\Theta^B \\ d\Xi \end{pmatrix} = \mathsf{N}_1 dy + \mathsf{N}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix} \tag{8}$$

where the matrices M, N_1 and N_2 are defined below.

The M matrix captures interactions between the goods purchases of the two household members and has the form

$$\mathsf{M} \equiv \begin{pmatrix} I & \mathsf{A} & 0 \\ \mathsf{B} & I & 0 \\ 0 & \mathsf{C} & I \\ \mathsf{D} & 0 & I \end{pmatrix}$$

where the non-zero blocks are given by

$$
\begin{aligned}
\mathsf{A} &= -\begin{pmatrix} f_y^A \\ F_y^A \end{pmatrix}\begin{pmatrix} p \\ P_B \end{pmatrix}' + \begin{pmatrix} 0 & f_{Q_B}^A \\ 0 & F_{Q_B}^A \end{pmatrix} \equiv \mathsf{A}_1 + \mathsf{A}_2, \\
\mathsf{B} &= -\begin{pmatrix} f_y^B \\ F_y^B \end{pmatrix}\begin{pmatrix} p \\ P_A \end{pmatrix}' + \begin{pmatrix} 0 & f_{Q_A}^B \\ 0 & F_{Q_A}^B \end{pmatrix} \equiv \mathsf{B}_1 + \mathsf{B}_2, \\
\mathsf{C} &= -G_y^A \begin{pmatrix} p \\ P_B \end{pmatrix}' + \begin{pmatrix} 0 & G_{Q_B}^A \end{pmatrix} \equiv \mathsf{C}_1 + \mathsf{C}_2 \\
\mathsf{D} &= -G_y^B \begin{pmatrix} p \\ P_A \end{pmatrix}' + \begin{pmatrix} 0 & G_{Q_A}^B \end{pmatrix} \equiv \mathsf{D}_1 + \mathsf{D}_2.
\end{aligned}
$$

The components A_1, B_1, C_1 and D_1 arise from interaction through the budget constraint, as greater purchases of any good individually contributed by one partner decreases the amount left over from the household budget for purchases by the other. These matrices each have rank 1, being each the outer product of a vector of income derivatives and a vector of prices.

The components A_2, B_2, C_2 and D_2 arise from the effect of one individual's purchases of individually contributed public goods on the preference ordering of the other over the goods individually contributed by the other. Such terms are generically of rank n_B, $\min(m+n_B, n_A)$, $\min(n_X, n_B)$ and $\min(n_X, n_A)$, respectively[4]. They disappear if either each person's preferences over their contributed goods are separable from the public goods individually contributed by the other or if all public goods are jointly contributed[5]. Since the special case in which, for whichever of these reasons, A_2, B_2, C_2 and D_2 disappear provides interesting simplifications of results, in what follows we will refer to it repeatedly as *contributory separability.*

Taking these observations together, we see that the matrices A, B, C and D are therefore of rank $1+n_B$, $\min(m+n_B, 1+n_A)$, $\min(n_X, 1+n_B)$ and $\min(n_X, 1+n_A)$,

[4]Each has only n_B, n_A, n_B and n_A non zero columns, respectively, corresponding to the number of public goods individually contributed by the other but in the case of B_2, C_2 and D_2 this determines the rank only if the numbers of rows $m+n_B$, n_X and n_X, respectively, are not short of n_A.

[5]The latter of these is the case considered in Lechene and Preston (2005)

The N_1 and N_2 matrices take the form

$$\mathsf{N}_1 \equiv \begin{pmatrix} f_y^A \\ F_y^A \\ f_y^B \\ F_y^B \\ G_y^A \\ G_y^B \end{pmatrix} \quad \text{and} \quad \mathsf{N}_2 \equiv \begin{pmatrix} f_p^A - f_y^A q^{B\prime} & f_{P_A}^A & -f_y^A Q_B' & f_R^A \\ F_p^A - F_y^A q^{B\prime} & F_{P_A}^A & -F_y^A Q_B' & F_R^A \\ f_p^B - f_y^B q^{A\prime} & -f_y^B Q_A' & f_{P_B}^B & f_R^B \\ F_p^B - F_y^B q^{A\prime} & -F_y^B Q_A' & F_{P_B}^B & F_R^B \\ G_p^A - G_y^A q^{B\prime} & G_{P_A}^A & -G_y^A Q_B' & G_R^A \\ G_p^B - G_y^B q^{B\prime} & -G_y^B Q_A' & G_{P_B}^B & G_R^A \end{pmatrix}$$

and are composed of conventional income and price effects, excepting that it is necessary to recognise in N_2 that increases in the prices of the public goods individually contributed by one partner decrease the amount left over from the household budget for purchases by the other.

The system (8) is overdetermined, specifically in the final $2n_X$ lines which imply alternative expressions for dX. With $n_X = 1$ compatibility is ensured by adding up, whereas for $n_X > 1$ similar issues arise to those discussed above concerning the nongenericity of such cases. If we let $\bar{\mathsf{M}}$, $\bar{\mathsf{N}}_1$ and $\bar{\mathsf{N}}_2$ denote the submatrices of M, N_1 and N_2 obtained by deleting the final n_X rows then we can rearrange to get

$$\begin{pmatrix} d\theta^A \\ d\Theta^A \\ d\theta^B \\ d\Theta^B \\ d\Xi \end{pmatrix} = \bar{\mathsf{M}}^{-1}\bar{\mathsf{N}}_1 dy + \bar{\mathsf{M}}^{-1}\bar{\mathsf{N}}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix}.$$

Since we work in terms of household purchases (q, Q_A, Q_B, X), we have therefore

$$\begin{pmatrix} d\theta \\ d\Theta^A \\ d\Theta^B \\ d\Xi \end{pmatrix} = \mathsf{E}\bar{\mathsf{M}}^{-1} \left(\bar{\mathsf{N}}_1 dy + \bar{\mathsf{N}}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \\ dR \end{pmatrix} \right)$$

where

$$\mathsf{E} \equiv \begin{pmatrix} I_m & 0 & I_m & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & 0 & I_{n_B+n_X} \end{pmatrix}$$

is an appropriate aggregating matrix.

The pseudo-Slutsky matrix now follows from:

$$\Psi = \mathsf{E}\bar{\mathsf{M}}^{-1}\left(\bar{\mathsf{N}}_2 + \bar{\mathsf{N}}_1 \begin{pmatrix} q \\ Q_A \\ Q_B \\ X \end{pmatrix}'\right) = \mathsf{E}\bar{\mathsf{M}}^{-1}\Phi$$

where

$$\begin{aligned} \Phi &= \begin{pmatrix} f_p^A + f_y^A q^{A\prime} & f_{P_A}^A + f_y^A Q_A' & 0 & f_R^A + f_y^A X' \\ F_p^A + F_y^A q^{A\prime} & F_{P_A}^A + F_y^A Q_A' & 0 & F_R^A + F_y^A X' \\ f_p^B + f_y^B q^{B\prime} & 0 & f_{P_B}^B + f_y^B Q_B' & f_R^B + f_y^B X' \\ F_p^B + F_y^B q^{B\prime} & 0 & F_{P_B}^B + F_y^B Q_B' & F_R^B + F_y^B X' \\ G_p^A + G_y^A q^{A\prime} & G_{P_A}^A + G_y^A Q_A' & 0 & G_R^A + G_y^A X' \end{pmatrix} \\ &= \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 & \Psi_{qX}^A \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 & \Psi_{QX}^A \\ \Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \\ \Psi_{Xq}^A & \Psi_{XQ}^A & 0 & \Psi_{XX}^A \end{pmatrix}. \end{aligned}$$

Note that the terms in Φ are all elements of the underlying symmetric and negative semidefinite true individual conditional Slutsky matrices corresponding to the individual decision problems

$$\Psi^A \equiv \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 & \Psi_{qX}^A \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 & \Psi_{QX}^A \\ 0 & 0 & 0_{n_B,n_B} & 0 \\ \Psi_{Xq}^A & \Psi_{XQ}^A & 0 & \Psi_{XX}^A \end{pmatrix} \text{ and } \Psi^B \equiv \begin{pmatrix} \Psi_{qq}^B & 0 & \Psi_{qQ}^B & \Psi_{qX}^B \\ 0 & 0_{n_A,n_A} & 0 & 0 \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B & \Psi_{QX}^B \\ \Psi_{Xq}^B & 0 & \Psi_{XQ}^B & \Psi_{XX}^B \end{pmatrix}.$$

Everything is now in place for the main result of the paper.

Theorem 2 *In an income pooling equilibrium, the pseudo-Slutsky matrix can be decomposed as*

$$\Psi = \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \Lambda$$

where

$$\begin{aligned} \operatorname{rank}(\Delta_1) &\leq \operatorname{rank}(\mathsf{B}) \leq \min(1 + n_A, m + n_B) \\ \operatorname{rank}(\Delta_2) &\leq \operatorname{rank}(\mathsf{A}) \leq 1 + n_B \\ \operatorname{rank}(\Lambda) &\leq n_X - 1 \end{aligned}$$

and

$$\operatorname{rank}(\Delta_1 + \Delta_2 + \Lambda) \leq n + \min(1, m - \max(n_A - n_B, 1)).$$

Proof of Theorem 2.

See the Appendix. ■

The individual Slutsky matrices Ψ^A and Ψ^B are symmetric and negative semidefinite and so therefore is their sum. The deviation from conventional demand properties is therefore determined by the properties of $\Delta_1 + \Delta_2 + \Lambda$. In the general case where the number of private goods is at least two and two partners contribute to roughly similar numbers of public goods then the rank of the departure from a negative and semidefinite pseudo-Slutsky matrix is therefore at most $n+1$, one more than the number of public goods. Provided that this condition holds, it is immaterial whether public goods are jointly or individually contributed since, say, an increase in n_X matched by a corresponding fall in $n_A + n_B$ simply results in offsetting changes in the ranks of Λ and $\Delta_1 + \Delta_2$.

Although the theorem establishes only a bound on the rank of the sum of the matrices $\Delta_1 + \Delta_2 + \Lambda$, inspection of the form of these matrices suggests that this bound will generically be attained[6].

To reduce the rank of the departure further requires reduction in the rank of either A or B and the most obvious restrictions to achieve this are separability restrictions. In particular, the case which we labelled contributory separability reduces A and B to rank 1 matrices and therefore in the generic case of $n_X = 1$ reduces the rank of the departure dramatically to 2, a small number but still one more than the same departure in the collective model[7] (Browning and Chiappori 1998). Intermediate cases corresponding to partial separability will give departures of intermediate rank.

[6] Generically, only a single public good is jointly contributed in income pooling equilibria in which case Λ disappears. The matrices Δ_1 and Δ_2 are both matrix products in which the factors of lowest rank are B and A respectively and therefore the rank of their sum will generically be the sum of the ranks of A and B. By "generically", we mean holding on an open set in the space of the couple's preferences, incomes and prices under some suitable topology.

[7] This is as observed in Lechene and Preston 2005, whose results are substantially generalised by Theorem 2.

4 Separate spheres

4.1 Separate spheres

In separate spheres equilibrium, by contrast, there are no jointly contributed public goods, $n_X = 0$ and partners contribute to disjoint subsets of public goods. The term is taken from Lundberg and Pollak (1993) who consider such a case as the threat point in a household bargaining model. All goods are individually contributed and household spending on each of the goods to which either partner contributes is constrained by their own individual income. Individual demands follow from solving

$$\max_{q^A, Q_A} u^A(q^A, Q_A, Q_B) \text{ s. t. } p'q^A + P_A' Q_A \leq y^A, \quad Q_A \geq 0,\ q^A \geq 0$$

and

$$\max_{q^B, Q_B} u^B(q^B, Q_A, Q_B) \text{ s. t. } p'q^B + P_B' Q_B \leq y^B, \quad Q_B \geq 0,\ q^B \geq 0.$$

and income pooling does not hold. Instead

$$q^A = f^A(y^A, p, P, Q_B) \tag{9}$$
$$Q_A = F^A(y^A, p, P, Q_B) \tag{10}$$
$$q^B = f^B(y^B, p, P, Q_A) \tag{11}$$
$$Q_B = F^B(y^B, p, P, Q_A) \tag{12}$$

and equilibrium demands depend on the distribution of income.

Assuming again uniqueness, we can write the quantities solving the system of equations (9) to (12) as functions $\tilde{\theta}^A(y^A, y^B, p, P)$, $\tilde{\Theta}^A(y^A, y^B, p, P)$, $\tilde{\theta}^B(y^A, y^B, p, P)$ and $\tilde{\Theta}^B(y^A, y^B, p, P)$ and household demands as

$$\tilde{\theta}(y^A, y^B, p, P) = \tilde{\theta}^A(y^A, y^B, p, P) + \tilde{\theta}^B(y^A, y^B, p, P),$$
$$\tilde{\Theta}(y^A, y^B, p, P) = \begin{pmatrix} \tilde{\Theta}^A(y^A, y^B, p, P) \\ \tilde{\Theta}^B(y^A, y^B, p, P) \end{pmatrix}.$$

Even to define the pseudo-Slutsky matrix in such a setting presents a problem as these equilibrium demands are not functions of household income.

We consider two alternative approaches to dealing with this. On the one hand we can define individual pseudo-Slutsky matrices, $\tilde{\Psi}^A$ and $\tilde{\Psi}^B$, by taking derivatives with respect to individual incomes[8]

$$\tilde{\Psi}^i \equiv \begin{pmatrix} \tilde{\theta}_p & \tilde{\theta}_P \\ \tilde{\Theta}_p & \tilde{\Theta}_P \end{pmatrix} + \begin{pmatrix} \tilde{\theta}_{y^i} \\ \tilde{\Theta}_{y^i} \end{pmatrix} \begin{pmatrix} \tilde{\theta} \\ \tilde{\Theta} \end{pmatrix}' \qquad i = A, B.$$

While having the merit of consistency with the noncooperative model under consideration, these are plainly not what would be computed by someone fitting a conventional unitary demand system to household data. To understand what such an exercise would produce, we need a way of calculating the derivative of household demands with respect to household income. To this end we define an income allocation function specifying individual incomes as a function of household income, prices, and any other relevant household characteristics[9] Z, $y^A = \phi(y, p, P, Z)$, $y^B = y - \phi(y, p, P, Z)$ so that we can write equilibrium household demands as

$$\begin{aligned} \theta(y, p, P, Z) &= \theta^A(y, p, P, Z) + \theta^B(y, p, P, Z) \\ &= \tilde{\theta}^A(\phi(y, p, P, Z), y - \phi(y, p, P, Z), p, P) \\ &\qquad + \tilde{\theta}^B(\phi(y, p, P, Z), y - \phi(y, p, P, Z), p, P) \end{aligned}$$

and

$$\begin{aligned} \Theta(y, p, P, Z) &= \begin{pmatrix} \Theta^A(y, p, P, Z) \\ \Theta^B(y, p, P, Z) \end{pmatrix} \\ &= \begin{pmatrix} \tilde{\Theta}^A(\phi(y, p, P, Z), y - \phi(y, p, P, Z), p, P) \\ \tilde{\Theta}^B(\phi(y, p, P, Z), y - \phi(y, p, P, Z), p, P) \end{pmatrix}. \end{aligned}$$

and continue to define a household pseudo-Slutsky matrix Ψ through (7). The empirical investigator needs only to estimate $\theta(y, p, P, Z)$ and $\Theta(y, p, P, Z)$

[8] We owe the idea to consider the properties of these individualised pseudo-Slutsky matrices to Martin Browning.

[9] Though we do not explore the role played by the variables Z they can be seen as playing a role analogous to the "distribution factors" in Browning and Chiappori 1998. Their presence in ϕ offers an interpretation for why characteristics not associated with preferences might be found to influence household demands in the current setting just as they do in the collective setting. The way in which they could do so would, as in that paper, clearly be heavily restricted, entering as they do only through the scalar function ϕ.

and is not required to know the form of the underlying function $\phi(y, p, P, Z)$ beyond conditioning on an appropriate set of potential distribution factors Z.

We thus have two different types of equilibrium demand functions and investigate the properties of both.

4.2 Adding up and homogeneity

Whilst it is obvious that adding-up continues to hold, homogeneity is no longer as simple.

Theorem 3 *In separate spheres equilibrium, household Nash equilibrium demands satisfy*

1. *(Adding up)*
   - $p'\theta(y, p, P) + P'\Theta(y, p, P) = y$
   - $p'\tilde{\theta}(y^A, y^B, p, P) + P'\tilde{\Theta}(y^A, y^B, p, P) = y^A + y^B$
2. *(Homogeneity)*
   - $\theta(\lambda y, \lambda p, \lambda P) = \theta(y, p, P)$ *and* $\Theta(\lambda y, \lambda p, \lambda P) = \Theta(y, p, P)$ *for any* $\lambda > 0$ *only if* $\phi(\lambda y, \lambda p, \lambda P, Z) = \lambda\phi(y, p, P, Z)$.
   - $\tilde{\theta}(\lambda y^A, \lambda y^B, \lambda p, \lambda P) = \tilde{\theta}(y^A, y^B, p, P)$ *and* $\tilde{\Theta}(\lambda y^A, \lambda y^B, \lambda p, \lambda P) = \tilde{\Theta}(y^A, y^B, p, P)$ *for any* $\lambda > 0$

Proof of Theorem 3.

1. Adding up of demands follows from the fact that the partners are on their individual budget constraints.
2. Homogeneity of $\tilde{\theta}$ and $\tilde{\Theta}$ follows from homogeneity of the individual demand functions but this guarantees homogeneity of θ and Θ only if individual incomes vary homogeneously with household incomes and prices.

■

4.3 Negativity and symmetry

As in the income pooling case, we substitute the equilibrium functions into (9) to (12) and differentiate, to derive, in this case, two systems of equations for equilibrium quantity responses.

Firstly, for the equilibrium demands conditioned on individual incomes,

$$\mathcal{M}\begin{pmatrix} d\tilde{\theta}^A \\ d\tilde{\Theta}^A \\ d\tilde{\theta}^B \\ d\tilde{\Theta}^B \end{pmatrix} = \mathcal{N}_1^A dy^A + \mathcal{N}_1^B dy^B + \mathcal{N}_2 \begin{pmatrix} dp \\ dP_A \\ dP_B \end{pmatrix} \tag{13}$$

and, secondly, for the system conditioned on household income,

$$\begin{aligned} \mathcal{M}\begin{pmatrix} d\theta^A \\ d\Theta^A \\ d\theta^B \\ d\Theta^B \end{pmatrix} &= \left[\phi_y \mathcal{N}_1^A + (1-\phi_y)\mathcal{N}_1^B\right] dy \\ &\quad + \left[\mathcal{N}_2 + (\mathcal{N}_1^A - \mathcal{N}_1^B)\begin{pmatrix} \phi_p \\ \phi_{P_A} \\ \phi_{P_B} \end{pmatrix}'\right]\begin{pmatrix} dp \\ dP_A \\ dP_B \end{pmatrix} \end{aligned} \tag{14}$$

where the matrices $\mathcal{M}$, $\mathcal{N}_1^A$, $\mathcal{N}_1^B$ and $\mathcal{N}_2$ are defined below.

The $\mathcal{M}$ matrix, common to the two systems, captures interactions between the goods purchases of the two household members and has the form

$$\mathcal{M} \equiv \begin{pmatrix} I & \mathcal{A} \\ \mathcal{B} & I \end{pmatrix}$$

where the off-diagonal blocks are given by

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} 0 & f_{Q_B}^A \\ 0 & F_{Q_B}^A \end{pmatrix}, \\ \mathcal{B} &= \begin{pmatrix} 0 & f_{Q_A}^B \\ 0 & F_{Q_A}^B \end{pmatrix} \end{aligned}$$

and capture only the effects of individually contributed public goods purchases on partners' preference orderings over their individually contributed goods. $\mathcal{A}$ and $\mathcal{B}$ are generically of rank n_B and $\min(n_A, m+n_B)$, respectively, and reduce to null matrices leaving $\mathcal{M}$ as simply an identity matrix in the case of contributory separability.

The $\mathcal{N}_1^A$, $\mathcal{N}_1^B$ and $\mathcal{N}_2$ matrices take the forms

$$\mathcal{N}_1^A \equiv \begin{pmatrix} f_y^A \\ F_y^A \\ 0 \\ 0 \end{pmatrix}, \quad \mathcal{N}_1^B \equiv \begin{pmatrix} 0 \\ 0 \\ f_y^B \\ F_y^B \end{pmatrix}, \quad \mathcal{N}_2 \equiv \begin{pmatrix} f_p^A & f_{P_A}^A & 0 \\ F_p^A & F_{P_A}^A & 0 \\ f_p^B & 0 & f_{P_B}^B \\ F_p^B & 0 & F_{P_B}^B \end{pmatrix}$$

and are composed of conventional income and price effects.

Defining a suitable aggregating matrix

$$\mathcal{E} \equiv \begin{pmatrix} I_m & 0 & I_m & 0 \\ 0 & I_{n_A} & 0 & 0 \\ 0 & 0 & 0 & I_{n_B} \end{pmatrix}$$

we can derive the pseudo-Slutsky matrices for the separate spheres case:

$$\begin{aligned}
\tilde{\Psi}^i &= \mathcal{E}\mathcal{M}^{-1} \left(\mathcal{N}_2 + \mathcal{N}_1^i \begin{pmatrix} q \\ Q_A \\ Q_B \end{pmatrix}' \right) \\
&= \mathcal{E}\mathcal{M}^{-1} \left[\Phi + (\mathcal{N}_1^A - \mathcal{N}_1^B)\zeta_i' \right] \qquad i = A, B \\
\Psi &= \mathcal{E}\mathcal{M}^{-1} \left(\left[\mathcal{N}_2 + (\mathcal{N}_1^A - \mathcal{N}_1^B) \begin{pmatrix} \phi_p \\ \phi_{P_A} \\ \phi_{P_B} \end{pmatrix}' \right] \right. \\
&\qquad \left. + \left[\phi_y \mathcal{N}_1^A + (1-\phi_y)\mathcal{N}_1^B \right] \begin{pmatrix} q \\ Q_A \\ Q_B \end{pmatrix}' \right) \\
&= \mathcal{E}\mathcal{M}^{-1} \left[\Phi + (\mathcal{N}_1^A - \mathcal{N}_1^B)\zeta' \right]
\end{aligned}$$

where

$$\Phi = \begin{pmatrix} \Psi_{qq}^A & \Psi_{qQ}^A & 0 \\ \Psi_{Qq}^A & \Psi_{QQ}^A & 0 \\ \Psi_{qq}^B & 0 & \Psi_{qQ}^B \\ \Psi_{Qq}^B & 0 & \Psi_{QQ}^B \end{pmatrix}$$

and

$$\zeta_A = \begin{pmatrix} q^B \\ 0 \\ Q^B \end{pmatrix}, \quad \zeta_B = \begin{pmatrix} q^A \\ Q^A \\ 0 \end{pmatrix}, \quad \zeta = \phi_y \zeta^A + (1-\phi_y)\zeta^B + \begin{pmatrix} \phi_p \\ \phi_{P_A} \\ \phi_{P_B} \end{pmatrix}.$$

As in the income pooling case, Φ is made up of components of the true conditional Slutsky matrices

$$\Psi^A \equiv \begin{pmatrix} \Psi^A_{qq} & \Psi^A_{qQ} & 0 \\ \Psi^A_{Qq} & \Psi^A_{QQ} & 0 \\ 0 & 0 & 0_{n_B,n_B} \end{pmatrix} \text{ and } \Psi^B \equiv \begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} \\ 0 & 0_{n_A,n_A} & 0 \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} \end{pmatrix}.$$

while ζ_A, ζ_B and ζ are vectors.

Thus the comparable result to Theorem 2 for the separate spheres case is as follows.

Theorem 4 *In a separate spheres equilibrium, the pseudo-Slutsky matrices can be decomposed as*

$$\begin{aligned} \Psi &= \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \mathcal{K} \\ \tilde{\Psi}^i &= \Psi^A + \Psi^B + \Delta_1^i + \Delta_2^i + \mathcal{K}^i \qquad i = A, B \end{aligned}$$

where

$$\begin{aligned} \operatorname{rank}(\Delta_1) = \operatorname{rank}(\Delta_1^i) &\leq \operatorname{rank}(\mathcal{B}) = \min(n_A, m + n_B) \\ \operatorname{rank}(\Delta_2) = \operatorname{rank}(\Delta_2^i) &\leq \operatorname{rank}(\mathcal{A}) = n_B \\ \operatorname{rank}(\mathcal{K}) = \operatorname{rank}(\mathcal{K}^i) &= 1, \qquad i = A, B \end{aligned}$$

and

$$\begin{aligned} \operatorname{rank}(\Delta_1 + \Delta_2 + \mathcal{K}) &= \operatorname{rank}(\Delta_1^i + \Delta_2^i + \mathcal{K}^i) \\ &\leq n + \min(1, m - \max(n_A - n_B, 1)), \qquad i = A, B. \end{aligned}$$

Proof of Theorem 4.

See the Appendix. ■

Perhaps somewhat remarkably, we find the rank of the departure from conventional demand properties to be exactly the same as in the income pooling case. The typical rank of the departure is again $n + 1$, one more than the number of public goods, a result which therefore applies to any type of equilibrium. It is therefore immaterial to the generic rank of the departure not only how many public goods are jointly contributed but indeed whether any are jointly contributed at all. As in the case of income pooling, one would expect this bound on the rank to be generically attained[10]. Again separability restrictions on preferences will reduce this rank and, in particular, complete contributory separability would in this case lead $\mathcal{A}$ and $\mathcal{B}$ to disappear, reducing the rank of the departure to 1, the same rank of departure as found in the collective model[11].

[10]The ranks of $\mathcal{K}$, $\mathcal{K}^A$ and $\mathcal{K}^B$ are always 1. The matrices Δ_1 and Δ_2, as in the income pooling case, are both matrix products and their sum will generically have rank equal to the sum of the ranks of $\mathcal{A}$ and $\mathcal{B}$. The same holds for the sums of Δ_1^A and Δ_2^A and of Δ_1^B and Δ_2^B.

[11]The fact that the rank reduction in this case is greater than under income pooling arises because, with no public goods being jointly contributed, contributory separability constitutes a more demanding restriction.

5 Empirical testing

Theorems 2 and 4 establish a common bound for the rank of the departure of the pseudo-Slutsky matrix Ψ from a symmetric and negative semidefinite matrix. This bound is $n+1$ (assuming $n_A - n_B$ is unknown). This section assesses the usefulness of this bound for testing either cooperative or noncooperative behaviour.

Note firstly that, unless $n = 0$ in which case noncooperative behaviour is efficient, then the departure under Nash equilibrium is greater than the rank 1 departure found under the collective model. Browning and Chiappori (1998) discuss how to test a rank 1 departure by testing the rank of $\Psi - \Psi'$. If such tests fail to reject a rank 1 departure for couples then the results above establish clearly that cooperative behaviour cannot be rejected against noncooperative behaviour for any number of public goods.

What, however, if cooperative behaviour is rejected? Is it possible to use properties of the pseudo-Slutsky matrix to test compatibility with non-cooperative behaviour of the sort analysed here? The first point to note in this respect is that the nature of the departure depends upon the number of public goods. This makes sense. In the cooperative case the rank of the departure is 1 because all interaction arises through the single dimension of the sharing rule. In the noncooperative case interaction arises through the public goods and it is natural that the rank of the departure should depend upon how many public goods there are. There is an important implication of this. Either one knows how many goods are publicly consumed in the household or one can only test noncooperative behaviour jointly with a hypothesis about the number of public goods.

Lemma 3 in the Appendix[12] establishes equivalence between departure from symmetry of no more than a given rank and an associated bound on the rank of the skew symmetric matrix $\Psi - \Psi'$. In particular, it shows that if $\Psi - \Psi'$ is known to have rank at most s then it is possible to write Ψ as the sum of a symmetric matrix and a matrix of rank at most r whenever $2r + 1 \geq s$. If demands satisfy adding up then $\begin{pmatrix} p \\ P \end{pmatrix}' \Psi = 0$ and if they satisfy homogeneity then $\begin{pmatrix} p \\ P \end{pmatrix}' \Psi' = 0$. Therefore $\begin{pmatrix} p \\ P \end{pmatrix}' (\Psi - \Psi') = 0$

[12]This result generalises Lemma 1 of Browning and Chiappori 1998 to cover departures of *any* rank.

and the rank of $\Psi - \Psi'$ cannot exceed one less than its dimension $n + m$. A departure from symmetry of rank at most $n+1$ is therefore restrictive only if $2n+3 < n+m-1$. It must therefore be the case that $m \geq n+5$ in order to test for noncooperative behaviour with n public goods. For example, to test for noncooperative behaviour with one public good requires at least 7 goods in total of which 6 are private. If this is so then the hypothesis is testable.

6 Conclusion

In this paper, we establish properties of demands in the Nash equilibrium with two agents and voluntarily contributed public goods. This noncooperative model is the polar case to the cooperative model of Browning and Chiappori (1998) within the class of those models based on individual optimisation. In reality, neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending behaviour and analysis of such extreme cases can be seen as a first step towards understanding of a more adequate model.

We show that the nature of the departure from unitary demand properties in household Nash equilibrium is qualitatively similar to that in collectively efficient models in that negativity and symmetry of compensated price responses is not guaranteed. The counterpart to the Slutsky matrix can be shown to depart from negativity and symmetry by a matrix of bounded rank but this rank typically exceeds that found in the collective model unless strong auxiliary restrictions are placed on preferences. This constitutes a testable restriction on household demand functions provided the number of private goods is large enough relative to the number of public goods. Future work will explore sufficient conditions for consistency with noncooperative equilibrium within the household.

Appendix

Lemma 1 *In an income pooling equilibrium,*

$$R'\mathsf{C} = \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} \qquad \textit{and} \qquad R'\mathsf{D} = \begin{pmatrix} p \\ P_A \end{pmatrix}' - \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B}.$$

Proof of Lemma 1.

By adding up,

$$\begin{pmatrix} p \\ P_A \\ R \end{pmatrix}' \begin{pmatrix} f_y^A \\ F_y^A \\ G_y^A \end{pmatrix} = \begin{pmatrix} p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} f_y^B \\ F_y^B \\ G_y^B \end{pmatrix} = 1$$

and

$$\begin{pmatrix} p \\ P_A \\ R \end{pmatrix}' \begin{pmatrix} f_{Q_B}^A \\ F_{Q_B}^A \\ G_{Q_B}^A \end{pmatrix} = \begin{pmatrix} p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} f_{Q_A}^B \\ F_{Q_A}^B \\ G_{Q_A}^B \end{pmatrix} = 0.$$

Hence

$$\begin{aligned} R'\mathsf{C} + \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} &= \left(R'G_y^A + \begin{pmatrix} p \\ P_A \end{pmatrix}' \begin{pmatrix} f_y^A \\ F_y^A \end{pmatrix} \right) \begin{pmatrix} p \\ P_B \end{pmatrix}' \\ &\quad + \begin{pmatrix} 0 & R'G_{Q_B}^A + \begin{pmatrix} p \\ P_A \end{pmatrix}' \begin{pmatrix} f_{Q_B}^A \\ F_{Q_B}^A \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} p \\ P_B \end{pmatrix}' \end{aligned}$$

and

$$\begin{aligned} R'\mathsf{D} + \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B} &= \left(R'G_y^B + \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} f_y^B \\ F_y^B \end{pmatrix} \right) \begin{pmatrix} p \\ P_A \end{pmatrix}' \\ &\quad + \begin{pmatrix} 0 & R'G_{Q_A}^B + \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} f_{Q_A}^B \\ F_{Q_A}^B \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} p \\ P_A \end{pmatrix}'. \end{aligned}$$

$\blacksquare$

Lemma 2 *In a separate spheres equilibrium,*

$$\begin{pmatrix} p \\ P_A \end{pmatrix}' \mathcal{A} = \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathcal{B} = 0.$$

Proof of Lemma 2.

By adding up,

$$\begin{pmatrix} p \\ P_A \end{pmatrix}' \begin{pmatrix} f^A_{Q_B} \\ F^A_{Q_B} \end{pmatrix} = \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} f^B_{Q_A} \\ F^B_{Q_A} \end{pmatrix} = 0$$

from which the result is immediate. ■

Proof of Theorem 2.

The matrix $\bar{\mathsf{M}}$ has a block lower triangular structure which helps in inversion. Specifically

$$\bar{\mathsf{M}}^{-1} = \begin{pmatrix} I + \mathsf{AB}(I-\mathsf{AB})^{-1} & -\mathsf{A}(I-\mathsf{BA})^{-1} & 0 \\ -\mathsf{B}(I-\mathsf{AB})^{-1} & I + \mathsf{BA}(I-\mathsf{BA})^{-1} & 0 \\ \mathsf{CB}(I-\mathsf{AB})^{-1} & -(\mathsf{C} + \mathsf{CBA}(I-\mathsf{BA})^{-1}) & I \end{pmatrix}$$

Thus

$$\mathsf{E}\bar{\mathsf{M}}^{-1}\Phi = \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \Lambda$$

where

$$\begin{aligned}
\Delta_1 &= \mathsf{E}\begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} \mathsf{B}(I-\mathsf{AB})^{-1} \begin{pmatrix} \Psi^A_{qq} & \Psi^A_{qQ} & 0 & \Psi^A_{qX} \\ \Psi^A_{Qq} & \Psi^A_{QQ} & 0 & \Psi^A_{QX} \end{pmatrix} \\
\Delta_2 &= \mathsf{E}\begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} \mathsf{A}(I-\mathsf{BA})^{-1} \begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} & \Psi^B_{qX} \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} & \Psi^B_{QX} \end{pmatrix} \\
\Lambda &= E\begin{pmatrix} 0_{m+n_A+n_B,m+n} \\ \Lambda_X \end{pmatrix} \\
\Lambda_X &= -\begin{pmatrix} \Psi^B_{Xq} & 0 & \Psi^B_{XQ} & \Psi^B_{XX} \end{pmatrix} \\
&\quad -\begin{pmatrix} \mathsf{C} + (\mathsf{CB}+\mathsf{D})\mathsf{A}(I-\mathsf{BA})^{-1} \end{pmatrix} \begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} & \Psi^B_{qX} \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} & \Psi^B_{QX} \end{pmatrix}.
\end{aligned}$$

The rank of Δ_1 cannot exceed the rank of B which is at most $\min(1+n_A, m+n_B)$ and that of Δ_2 cannot exceed the rank of A which is at most $1+n_B$, each being defined as products involving these matrices.

The rank of Λ cannot exceed n_X since it contains only n_X non-zero rows. From Lemma 1,

$$\begin{aligned}
R'(\mathsf{C}+(\mathsf{CB}+\mathsf{D})\mathsf{A}(I-\mathsf{BA})^{-1}) &= \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} \\
+\left(\begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B} - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{AB} + \begin{pmatrix} p \\ P_A \end{pmatrix}' - \begin{pmatrix} p \\ P_B \end{pmatrix}' \mathsf{B}\right) \mathsf{A}(I-\mathsf{BA})^{-1} & \\
&= \begin{pmatrix} p \\ P_B \end{pmatrix}' - \begin{pmatrix} p \\ P_A \end{pmatrix}' \mathsf{A} + \begin{pmatrix} p \\ P_A \end{pmatrix}' (I-\mathsf{AB})\,(I-\mathsf{AB})^{-1}\mathsf{A} \\
&= \begin{pmatrix} p \\ P_B \end{pmatrix}'
\end{aligned}$$

and therefore

$$\begin{aligned}
R'\Lambda_X &= -R'\begin{pmatrix} \Psi^B_{Xq} & 0 & \Psi^B_{XQ} & \Psi^B_{XX} \end{pmatrix} - \begin{pmatrix} p \\ P_B \end{pmatrix}' \begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} & \Psi^B_{qX} \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} & \Psi^B_{QX} \end{pmatrix} \\
&= 0 \qquad (15)
\end{aligned}$$

by standard properties of the Slutsky matrix. Therefore the rank of Λ_X cannot exceed $n_X - 1$ and neither therefore can that of Λ.

The rank of $\Delta_1 + \Delta_2 + \Lambda$ cannot be greater than the sum of their ranks considered individually which is $n+\min(1, m-n_A+n_B)$. This number cannot exceed the dimension $n+m$ of the (square) matrix Ψ but can equal it in the one case $m = 1$ and $n_A = n_B$. In this case it becomes relevant that Δ_1, Δ_2 and Λ share a common linear dependency since, from Lemma 1,

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E} \begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} \mathsf{A} \\ -I \\ \mathsf{C} \end{pmatrix} = 0$$

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E} \begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \begin{pmatrix} -I \\ \mathsf{B} \\ \mathsf{D} \end{pmatrix} = 0$$

and, from (15),

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathsf{E}\Lambda = \begin{pmatrix} p \\ P_A \\ p \\ P_B \\ R \end{pmatrix}' \Lambda = R'\Lambda_X = 0.$$

This means that their sum cannot be of full rank and the maximum rank is reduced by 1 in this instance. (This is simply a consequence of adding up. Since household demands must satisfy the household budget constraint, Engel and Cournot aggregation must still hold for θ and Θ and Ψ must be singular as are Ψ^A and Ψ^B.)

The rank of the departure is therefore bounded from above by $n + \min(1, m - \max(n_A - n_B, 1))$. ■

Proof of Theorem 4.

The inverse of $\mathcal{M}$ has the form

$$\mathcal{M}^{-1} = \begin{pmatrix} I + \mathcal{AB}(I - \mathcal{AB})^{-1} & -\mathcal{A}(I - \mathcal{BA})^{-1} \\ -\mathcal{B}(I - \mathcal{AB})^{-1} & I + \mathcal{BA}(I - \mathcal{BA})^{-1} \end{pmatrix}$$

Thus

$$\begin{aligned} \Psi &= \Psi^A + \Psi^B + \Delta_1 + \Delta_2 + \mathcal{K}, \\ \tilde{\Psi}^i &= \Psi^A + \Psi^B + \Delta_1^i + \Delta_2^i + \mathcal{K}^i \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= \mathcal{E}\begin{pmatrix} \mathcal{A} \\ -I \end{pmatrix} \mathcal{B}(I - \mathcal{AB})^{-1} \left[\begin{pmatrix} \Psi^A_{qq} & \Psi^A_{qQ} & 0 \\ \Psi^A_{Qq} & \Psi^A_{QQ} & 0 \end{pmatrix} + \mathcal{N}_1^A \zeta' \right] \\ \Delta_1^i &= \mathcal{E}\begin{pmatrix} \mathcal{A} \\ -I \end{pmatrix} \mathcal{B}(I - \mathcal{AB})^{-1} \left[\begin{pmatrix} \Psi^A_{qq} & \Psi^A_{qQ} & 0 \\ \Psi^A_{Qq} & \Psi^A_{QQ} & 0 \end{pmatrix} + \mathcal{N}_1^A \zeta_i' \right] \\ \Delta_2 &= \mathcal{E}\begin{pmatrix} -I \\ \mathcal{B} \end{pmatrix} \mathcal{A}(I - \mathcal{BA})^{-1} \left[\begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} \end{pmatrix} - \mathcal{N}_1^B \zeta' \right] \\ \Delta_2^i &= \mathcal{E}\begin{pmatrix} -I \\ \mathcal{B} \end{pmatrix} \mathcal{A}(I - \mathcal{BA})^{-1} \left[\begin{pmatrix} \Psi^B_{qq} & 0 & \Psi^B_{qQ} \\ \Psi^B_{Qq} & 0 & \Psi^B_{QQ} \end{pmatrix} - \mathcal{N}_1^B \zeta_i' \right] \end{aligned}$$

and

$$\mathcal{K} = \left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] \zeta' \qquad \mathcal{K}^i = \left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] \zeta_i'.$$

The rank of Δ_1 and Δ_1^i, $i = A, B$, cannot exceed the rank of $\mathcal{B}$ which is at most $\min(n_A, m + n_B)$ and that of Δ_2 and Δ_2^i, $i = A, B$, cannot exceed the rank of $\mathcal{A}$ which is at most n_B, each being defined as products involving these matrices. Moreover, $\mathcal{K}$ and $\mathcal{K}^i$, $i = A, B$, being matrix products involving an outer product of vectors[13], have rank 1.

The rank of $\Delta_1 + \Delta_2 + \mathcal{K}$ and of $\Delta_1^i + \Delta_2^i + \mathcal{K}^i$, $i = A, B$, cannot be greater than the sum of the ranks of the component matrices considered individually which is, in each case, $n + \min(1, m - n_A + n_B)$. This number cannot exceed the dimension $n + m$ of the (square) matrix Ψ but can equal it in the one case $m = 1$ and $n_A = n_B$. In this case it becomes relevant that Δ_1, Δ_2, $\mathcal{K}$, Δ_1^i, Δ_2^i and $\mathcal{K}^i$, $i = A, B$, all share a common linear dependency since, from Lemma 2,

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathcal{E} \begin{pmatrix} \mathcal{A} \\ -I \end{pmatrix} \mathcal{B} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \end{pmatrix}' \begin{pmatrix} \mathcal{A} \\ -I \end{pmatrix} \mathcal{B} \;=\; -\begin{pmatrix} p \\ P_B \end{pmatrix}' \mathcal{B} = 0$$

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathcal{E} \begin{pmatrix} -I \\ \mathcal{B} \end{pmatrix} \mathcal{A} = \begin{pmatrix} p \\ P_A \\ p \\ P_B \end{pmatrix}' \begin{pmatrix} -I \\ \mathcal{B} \end{pmatrix} \mathcal{A} \;=\; -\begin{pmatrix} p \\ P_A \end{pmatrix}' \mathcal{A} = 0$$

and, by adding up,

$$\begin{pmatrix} p \\ P \end{pmatrix}' \mathcal{E} \left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] = \begin{pmatrix} p \\ P_A \\ p \\ P_B \end{pmatrix}' \left[\mathcal{N}_1^A - \mathcal{N}_1^B\right] = 1 - 1 = 0.$$

Thus the maximum rank is reduced by 1 in this instance and the rank of the departure is therefore bounded from above by $n + \min(1, m - \max(n_A - n_B, 1))$. ∎

Lemma 3 *Let Ψ be a real $k \times k$ matrix such that the rank of $\Psi - \Psi'$ cannot exceed s. Then Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r if and only if either (i) $2r + 1 \geq s$ or (ii) $2r + 1 < s$ and $\Psi - \Psi'$ has rank at most $2r$.*

[13] By adding up, it is impossible for either $\mathcal{N}_1^A$ or $\mathcal{N}_1^B$ to be zero vectors so these matrices have rank of exactly 1.

Proof of Lemma 3.

For any real $k \times k$ matrix Ψ the matrix $\Psi - \Psi'$ is skew symmetric and therefore has even rank. If the rank of $\Psi - \Psi'$ cannot exceed s then its rank is therefore at most s if s is even and at most $s-1$ if s is odd.

Suppose Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r. Then

$$\Psi = S + \sum_{i=1}^{r} u_i v_i'$$

where $S = S'$ and u_i, v_i are $k \times 1$ vectors, $i = 1, \ldots, r$. Then

$$\Psi - \Psi' = \sum_{i=1}^{r} (u_i v_i' - v_i u_i')$$

which has rank at most $2r$. If s is even then the rank is therefore at most $\min(2r, s)$ whereas if s is odd then the rank is therefore at most $\min(2r, s-1)$. In each case the bound of $2r$ is restrictive only if $2r + 1 < s$.

Conversely, suppose $\Psi - \Psi'$ has rank at most $2r$. (Note that this holds for any matrix Ψ if the rank of $\Psi - \Psi'$ cannot exceed s and $2r+1 \geq s$.) Since $\Psi - \Psi'$ is real and skew symmetric, it is possible (see, for example, Theorem 2.5 in Thompson 1988) to write $\Psi - \Psi' = ULU'$ for some orthogonal matrix U and a block diagonal matrix $L = \mathrm{diag}(L_1, \ldots, L_r, 0, \ldots, 0)$ where

$$L_i = \begin{pmatrix} 0 & \lambda_i \\ -\lambda_i & 0 \end{pmatrix}$$

for some real λ_i, $i = 1, \ldots, r$. Therefore $\Psi - \Psi' = \sum_{i=1}^{r} \lambda_i (u_i v_i' - v_i u_i')$ where $U = (u_1 \; v_1 \; u_2 \; v_2 \; \ldots)$. Then $\Psi - \sum_{i=1}^{r} \lambda_i u_i v_i' = \Psi' - \sum_{i=1}^{r} \lambda_i v_i u_i'$ is symmetric. Call this matrix S. Then Ψ can be written as the sum of a symmetric matrix and a matrix of rank at most r

$$\Psi = S + \sum_{i=1}^{r} \lambda_i u_i v_i'.$$

■

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