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Monotonicity of bargaining solutions in relative disagreement utility

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Monotonicity of Bargaining Solutions in Relative Disagreement Utility

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Abstract

Does a party benefit from taking an action which worsens the disagreement point for all parties, but relatively less for themselves? This question has important applications to areas including antitrust, household bargaining, trade negotiations and conflict. I define and study a desirable property of bargaining solutions, which I call "monotonicity in relative disagreement utility" (MRDU). MRDU requires that, if a party's disagreement utility increases (decreases) more (less) than any other player's, then their bargaining outcome must strictly improve. The Egalitarian solution is the only cooperative bargaining solution which guarantees MRDU. Other solutions, including the Nash, Kalai-Smorodinsky and non-Egalitarian Proportional solutions, may violate this property under a general bargaining set. Under transferable utility, a solution guarantees MRDU if and only if it belongs to a class of solutions which satisfies four widely used axioms. This class includes the Egalitarian, Nash and Kalai-Smorodinsky solutions.

Keywords: cooperative bargaining; disagreement point; solution properties; Egalitarian solution.

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1 Introduction

Consider a negotiation between multiple parties. If the disagreement point changes, does the party with the bigger increase (smaller decrease) in disagreement utility benefit, and at least one of their opponents lose out? This question is empirically relevant in several contexts. For instance, in an antitrust context, if a firm leverages their market power to worsen the disagreement point both for themselves and for a firm they are bargaining with, in a manner which has a larger impact on the other firm, will this conduct have the effect of improving their own bargaining outcome?

For example, in recent US and UK antitrust litigation against IP holder and chipset producer Qualcomm, a core question was whether Qualcomm's alleged "no licence no chips" (NLNC) policy distorted bargaining between Qualcomm and handset manufacturers in a manner which inflated royalties received by Qualcomm.[1][2] The alleged NLNC policy consisted in the threat that Qualcomm would not sell chipsets to handset manufacturers if the parties failed to reach a licensing agreement on certain cellular communications patents held by Qualcomm. Since both parties would benefit from the exchange of chipsets, the mechanism at play involved a worsening of the disagreement point in licensing negotiations for both. However, plaintiffs argued that this worsening was more severe for handset manufacturers.[3] The case made by the plaintiffs was that Qualcomm anti-competitively leveraged its alleged market power in chipsets to strengthen its bargaining position in licensing negotiations, allegedly leading to inflated royalties. This argument implicitly relies on the assumption that a conduct which worsens the disagreement point for both parties in a negotiation, but relatively more for the other party, will lead to a more favourable outcome for oneself.

A closely related question in household economics is whether someone would gain from taking an action to make the disagreement point in household negotiations

[1] See United Kingdom Competition Appeal Tribunal *In re: Consumers' Association v Qualcomm Incorporated, Transcript of Case Management Conference (Day 1)*, Case No. 1382/7/7/21 (2024, July); United States District Court Northern District of California San Jose Division *In re: Qualcomm Antitrust Litigation, Federal Trade Commission's Complaint for Equitable Relief*, Case No. 5:17-cv-00220 (2017, January).

[2] In the US proceedings, the 2019 District Court judgment against Qualcomm was later reversed by the Ninth Circuit. In the UK, the proceedings ended in a drop-hands settlement.

[3] United Kingdom Competition Appeal Tribunal *In re: Consumers' Association v Qualcomm Incorporated, Transcript of Trial (Day 10)*, Case No. 1382/7/7/21 (2025, October).

more costly both for themselves and for their partner, but more substantially so for their partner. More broadly, we may wish to predict which partner will benefit from a policy change, or any situation, which has an asymmetric impact on the disagreement point (potentially divorce) for the two partners. Examples of such situations include policies which give more favourable tax treatment to married than unmarried couples, the joint ownership of a substantial and illiquid asset, or the presence of children. For instance, if couples are formed between people of two "types", and while both prefer their children to grow up without experiencing their parents divorcing, one of the two types has a stronger preference in this regard, does having children worsen the bargaining outcome for the parent with the stronger preference?

In this paper, I build a theoretical foundation to assess these questions. I define a desirable property of bargaining solutions, which I call "monotonicity in relative disagreement utility" (MRDU). While this property has not previously been defined or analysed, it captures what seems to be a widespread intuition about negotiations: that if one party's disagreement utility is affected favourably compared to others', then the former will benefit. This type of thinking is common in academic economics too, for instance in the household economics literature, and seems to capture a relevant empirical feature of negotiations.

For example, Gu et al. (2024) study the impact of spousal differences in characteristics, including earnings, education and cognitive ability, on a spouse's say in household financial decisions. This is measured as the alignment between that spouse's risk preferences and the risk preferences implied by the household's investment portfolio, in a context where spouses often have different risk preferences from each other. They find that a larger gap between male and female characteristics predicts a significantly larger gap in influence over financial decisions. Their paper is grounded in a household bargaining model where the disagreement point is influenced by these characteristics.[4] This is in line with a large proportion of the household economics literature, which models the disagreement point as divorce, meaning that disagreement utility is affected by characteristics influencing e.g. one's financial security or one's ease of re-marrying post-divorce. Focusing on the impact of changes in *differences in characteristics between spouses* suggests that if one spouse's disagreement utility increases

[4] A large share of the household economics literature relies on the collective model of decision-making, which requires households to bargain cooperatively and reach Pareto efficient outcomes, while remaining agnostic about the specific bargaining solution employed in negotiations. See chapter 3.5 of Browning et al. (2014).

more than the other then the former will have an improved bargaining outcome.

Similarly, Oreffice (2011) finds that bargaining outcomes of both homosexual and heterosexual couples respond substantially to shifts in bargaining power, measured by *differences* in non-labour-income and age between partners. Bertocchi et al. (2014) study the impact of *differences in characteristics* of wife and husband (income, education, age) and find that a relative improvement for the wife increases the probability of her being responsible for the household's economic and financial choices. Similarly, Aizer (2010) finds that a decrease in the pay *gap* (male wage minus female wage) between partners reduces domestic violence. Interestingly, Aizer (2010) reports similar findings when using the wage ratio, which is also frequently treated as a determinant of intra-household bargaining power (as well as the sex ratio), and found to have significant impacts empirically, in household economics (e.g. see Chapter 1 of Browning et al. (2014), Lise and Yamada (2019), Chiappori, Fortin and Lacroix (2002)). Ratios of characteristics are not the same as differences in characteristics, but the ratio approach also relies on, and empirically confirms, the same high-level intuition that *relative* changes in disagreement utility matter to outcomes.

While MRDU is relevant to several important questions, the academic literature to date has not formally defined this property, analysed it, explored which bargaining solutions guarantee it, or characterised possible violations under standard solutions. By addressing this gap, this paper contributes to the well-established literature on cooperative bargaining solutions and their properties. Within that literature, a paper closely related to this one is Thomson (1987), which shows that the Nash solution, the Kalai-Smorodinsky solution, and the Egalitarian solution all satisfy a condition termed "disagreement point monotonicity". Disagreement point monotonicity requires that if a player's disagreement utility improves, while others' remain constant, then that player's outcome must weakly improve. Thomson (1987) also shows that the Nash solution and the Kalai-Smorodinsky solution fail "strong disagreement point monotonicity", while the Egalitarian solution satisfies it. This is the requirement that, if a player's disagreement utility improves, while others remain constant, then all other players' outcomes must weakly decrease.

Another closely related paper is Bossert (1994), which investigates the effect of an increase in one player's disagreement utility, combined with a decrease in another player's disagreement utility, while all other players' disagreement utilities remain constant. Bossert (1994) shows that only the Egalitarian bargaining solution satisfies "strong transfer responsiveness". This property requires that the outcome must weakly improve for the player with increased disagreement utility, weakly worsen for the player with decreased disagreement utility, and that the impact on other players must be bounded by the magnitude of the impact on the players whose disagreement utility changed.

While Thomson (1987) only speaks to scenarios in which the whole bargaining problem is unchanged apart from one player's disagreement utility, and Bossert (1994) focuses on a case where only one player's disagreement utility improves and only one player's worsens, this paper turns to more general changes in the disagreement point, including all players' disagreement utilities changing, and more than one player's disagreement utility changing in the same direction. There is no direct relationship between strong transfer responsiveness and MRDU, so that the results in Bossert (1994) cannot be used to infer those in this paper, and vice versa. The same is true for the strong disagreement point monotonicity of Thomson (1987). Instead, the disagreement point monotonicity of Thomson (1987) is a weaker condition than MRDU, so that satisfying MRDU guarantees satisfying disagreement point monotonicity, but not the other way round.

I show that, notwithstanding its intuitive appeal, MRDU is guaranteed to hold only in a limited number of settings. Under a general bargaining set, MRDU is not guaranteed by the two most widely used cooperative bargaining solutions, the Nash solution and the Kalai-Smorodinsky solution. This is because MRDU requires cardinal unit comparability (interpersonal comparability of utility changes) and therefore is inconsistent with the invariance assumption made by both of those solutions. However, both the Nash and Kalai-Smorodinsky solutions always satisfy MRDU under the restriction of a transferable utility (TU) bargaining set, which effectively forces cardinal unit comparability. Instead, the non-Egalitarian Proportional solution never guarantees MRDU, because MRDU requires a symmetric treatment of players. Instead, the Egalitarian solution, which is consistent both with cardinal unit comparability and symmetry, always satisfies MRDU.

One might question whether it is worth studying properties, like MRDU or like the strong transfer responsiveness of Bossert (1994), which are inconsistent with invariance to positive affine transformations. While invariance is a theoretically appealing axiom, we also have good reasons to question whether it is always appropriate. For instance, Myerson (1977) shows that four natural axioms about cooperative bargaining entail interpersonal comparisons in utility gains, Nydegger and Owen (1974) provide experimental evidence suggesting people make interpersonal utility comparisons when bargaining, and Harrison and Rutström (1991) argue that trade negotiators make interpersonal comparisons of gains. Politicians and commentators regularly make claims, in lay terms, about the likely outcomes of trade negotiations and conflicts as a function of interpersonal comparisons of disutility from disagreement, arguing that continued war is worse for one party than another, or that one country needs a trade deal more than another. For example, during Brexit negotiations, news headlines such as 'Europe really doesn't need us as much as we need them' (Goodall 2018, August 6) were common in the UK. Setting aside the question of which form of invariance axiom, if any, is appropriate in different contexts, MRDU's own intuitive appeal, its implicit use in the academic literature, and its apparent empirical fit in contexts such as household negotiations, provide a strong motivation to be interested in this property.

I show that the Egalitarian solution is the unique cooperative bargaining solution that guarantees MRDU under a general bargaining set. This adds to the existing literature, including Thomson (1987) and Bossert (1994), suggesting that the Egalitarian solution guarantees intuitive and empirically relevant properties that can be violated by the more commonly used Nash and Kalai-Smorodinsky solutions. I also show that, under TU, the Egalitarian solution coincides with a class of solutions satisfying a set of four commonly used axioms. Under TU, this class, which includes the Egalitarian, Nash and Kalai-Smorodinsky solutions, is the unique set of solutions that guarantees MRDU.

My findings suggest that, in contexts where TU is considered a plausible assumption, it is safe to assume MRDU. TU is a widely made assumption in bargaining theory (see Bergstrom and Varian (1985)), and TU bargaining sets are a widely studied subset of well-behaved bargaining sets, e.g. see Gul (1989). Moreover, TU has been frequently assumed in the broader economics literature (see Chiappori and Gugl (2020)), including in the context of household bargaining, e.g. see chapter 7 of Browning et al. (2014), and Chiappori, Dias and Meghir (2018).[5] In some contexts, TU is a descriptively realistic assumption, for instance in negotiations between two firms of similar

[5]Note that some papers define TU in the broader sense of a linear Pareto frontier with a fixed 'exchange rate' between utilities, which is not necessarily 1. Here, I use the more standard definition of TU, i.e. I use the term only in the case where that 'exchange rate' of utilities is 1, and 'non-TU linear Pareto frontier' for the case of a non-1 but fixed 'exchange rate' in utilities.

size, facing similar taxation, and not encumbered by pressing cash constraints. In this type of context, monetary payoffs can be transferred between firms, and there is no reason to believe the firms have substantially different valuations of those monetary units. In other contexts, such as household bargaining, TU may not be fully descriptively realistic, but it may be considered a useful simplification nonetheless.

Where TU is not viewed as an acceptable assumption, my findings suggest that MRDU is safe to assume only under the Egalitarian solution. Under other solutions, such as Nash, Kalai-Smorodinsky, and non-Egalitarian Proportional solutions, further consideration of the specifics is required. A solution not guaranteeing MRDU does not in itself speak to whether the solution is likely, or unlikely, to lead to MRDU violations, and in what circumstances. This is of course an extremely important question for empirical applications. For example, a solution which can violate MRDU, but in practice typically does not do so, might be a convenient modelling assumption in a context where MRDU is known to be a feature of negotiations. Equally, it may be sensible to assume MRDU in a context where standard bargaining solutions are theoretically capable of, but unlikely to, give rise to MRDU violations.

To provide some insight into these types of questions, I focus on a simplified setting with two players and a non-TU linear Pareto frontier. In this setting, I consider what changes in disagreement point give rise to MRDU violations under the Nash, Kalai-Smorodinsky and non-Egalitarian Proportional solutions. I find that MRDU is not violated as long as there is a sufficiently large difference between the change in disagreement utility for the party who was affected more favourably and the one who was affected less favourably. In the case of the Nash and Kalai-Smorodinsky solutions, this has to be sufficiently large relative to the slope of the Pareto frontier. Intuitively, this is because MRDU violations in this context are caused by asymmetries in the rate at which the utility of one party can be transformed into utility of the other party, so that MRDU is satisfied if and only if the asymmetry in the disagreement utility change is larger than the asymmetry in utility 'exchange rate'. When the slope is minus one, this coincides with the TU case and no violations of MRDU are possible.

In the case of the non-Egalitarian Proportional solution, MRDU is not violated as long as there is a sufficiently large difference between the larger and the smaller increase in disagreement utility relative to the asymmetry in proportionality weights assigned to players. Intuitively, this is because MRDU violations in this case are due to the asymmetric treatment of players. For proportionality weights which are close to being symmetric, the solution approximates the Egalitarian solution, which guarantees MRDU. For all three of the Nash, Kalai-Smorodinsky and non-Egalitarian Proportional solutions, MRDU is guaranteed with two players if the disagreement utility of one player increased and the disagreement utility of the other player decreased.

This paper is organised as follows. In section 2, I set out the bargaining framework. In section 3, I consider what class of bargaining solutions guarantees MRDU. In section 4, I analyse when the Nash solution guarantees MRDU, and when it violates it. In section 5, I do the same for the Kalai-Smorodinsky solution, and in section 6 for the non-Egalitarian Proportional solution. In section 7, I compare the nature and share of possible disagreement point changes that lead to MRDU violations for different bargaining solutions. I conclude in section 8.

2 Framework

2.1 The bargaining problem

An n-person bargaining problem is defined by a disagreement point d and a non-negative bargaining set $S \subseteq \mathbb{R}^n_+$. The disagreement point $d \in S$ specifies the outcome if players fail to reach an agreement. The bargaining set specifies all feasible combinations of player payoffs from coming to some agreement. As is standard, I assume this set is bounded, closed and convex, and contains at least one point which is strictly preferable to the disagreement point for all players.

I additionally assume that the bargaining set satisfies strict comprehensiveness, defined in a similar way as in Bossert (1994). Under comprehensiveness:

$$(x \in S) \wedge (0 \leq y \leq x) \wedge (x \neq y) \rightarrow (y \in S)$$

Additionally, under *strict* comprehensiveness:

$$(x \in S) \wedge (0 \leq y \leq x) \wedge (x \neq y) \rightarrow (\exists z \in S : z > y)$$

Strict comprehensiveness is stronger than free disposal, and also ensures the Pareto frontier is strictly decreasing. This is helpful to ensure that any ray drawn from the disagreement point at an angle $(0°, 90°)$ will intersect the boundary of the set at a

Pareto efficient point.

As an example of the relevance of strict comprehensiveness, imagine that we do not require it, so that a rectangular bargaining set is allowed. With a rectangular bargaining set, there is a unique Pareto efficient point (the top-right corner), so any bargaining solution requiring Pareto efficiency must select that point. In this case, no change in disagreement point will alter the solution. Hence, the party with the most favourable change in disagreement utility can never strictly gain from the change, while they trivially weakly gain.

2.1.1 Linear Pareto frontier

The bargaining set may be characterised by a linear Pareto frontier. In this case there is a fixed 'exchange rate' between utilities of different parties:

$$S = \left\{ u : 0 \leq u_i \leq \frac{\Gamma}{\beta_j}, \sum_j \beta_j u_j \leq \Gamma \right\}$$

The "size of the pie" Γ and the utility weights (utility 'exchange rate') β can be jointly normalised without loss of generality, so that the weights sum to 1. The normalised bargaining set is:

$$S = \left\{ u : 0 \leq u_i \leq \frac{T}{\eta_j}, \sum_j \eta_j u_j \leq T \sum_j \eta_j = 1 \right\}$$

The surplus R to be divided in negotiations is:

$$R = T - \sum_j \eta_j d_j$$

2.1.2 Transferable utility (TU)

Transferable utility (TU) is a special case of a linear Pareto frontier, where the utility weights in the equation of the frontier are all the same (the utility 'exchange rate' is 1). Note that, since utility weights cannot differ across individuals, TU is inconsistent with invariance to positive affine transformations (inconsistent with INV, but consistent with CSA-INV - see table 1). This is because applying different rescalings of utility to different players breaks the one-to-one relationship between utilities inherent in the TU setting.

Under TU, the bargaining set is defined as:

$$S = \left\{ u : 0 \leq u_i \leq T, \sum_j u_j \leq T \right\}$$

Where T is the maximum utility attainable. We can define the surplus from negotiations as:

$$R = T - \sum_j d_j$$

2.2 Bargaining axioms

This paper focuses on cooperative bargaining solutions, derived axiomatically. Commonly used axioms for cooperative bargaining solutions are listed in table 1, and are referred to throughout the paper. These axioms needn't be psychologically accurate descriptions of considerations that are explicitly made in negotiations, but properties that are required for agreements to be stable and/or which capture empirical regularities in bargaining outcomes. Several papers have provided non-cooperative game theoretic foundations for cooperative bargaining solutions, for instance Nash (1953), Rubinstein (1982), Binmore et al. (1986) and Carlsson (1991) for the Nash solution, Moulin (1984) for the Kalai-Smorodinsky solution and Bossert and Tan (1995), Dutta (2012) and Dutta (2024) for the Proportional class of solutions, including the Egalitarian solution as a special case.

Axiom	Definition
IR: individual rationality	IR (weak IR) requires that the solution is strictly (weakly) preferred to the disagreement point by all parties: $u_i^* > d_i, \forall i \ \left(u_i^* \geq d_i, \forall i\right)$
PAR: Pareto efficiency	No point in the bargaining set Pareto dominates the solution: $\neg\exists x \in S : (x \geq u^*) \wedge \left(\exists i : x_i > u_i^*\right)$
SYM: symmetry	If the bargaining problem is symmetric in that S and d are symmetric, then so is the solution: $u_i^* = x, \forall i$
INV: invariance to equivalent utility representations	Each player's utility can be transformed by a different positive affine transformation, and the resulting outcome will be the same as the original one, but with each utility transformed by that player-specific transformation: $u_i' = a_i u_i + b_i \rightarrow u_i^{*\prime} = a_i u_i^* + b_i, a_i > 0$. This precludes interpersonal utility comparisons.
O-INV: origin invariance	Each player's utility can be transformed by summing a different constant, and the resulting outcome will be the same as the original one, but with each utility transformed by that player-specific transformation: $u_i' = u_i + b_i \rightarrow u_i^{*\prime} = u_i^* + b_i$.
CS-INV: common scale invariance	Each player's utility can be transformed by multiplying it by the same positive constant, and the resulting outcome will be the same as the original one, but with each utility transformed by that transformation: $u_i' = a u_i \rightarrow u_i^{*\prime} = a u_i^*, a > 0$. This is consistent with full interpersonal comparability.
CSA-INV: common scale affine invariance	This is the combination of O-INV and CS-INV: $u_i' = a u_i + b_i \rightarrow u_i^{*\prime} = a u_i^* + b_i, a > 0$. This is the strongest invariance assumption that is consistent with interpersonal comparisons of utility gains.
IIA: independence of irrelevant alternatives	If the bargaining problem is unchanged excepting the removal of some utility combinations from the bargaining set, none of which are the original solution or the disagreement point, then the bargaining solution is unchanged: $((S' \subset S) \wedge (u^* \in S') \wedge (d' = d) \wedge (d \in S')) \rightarrow (u^{*\prime} = u^*)$
I-MON: individual monotonicity	If the bargaining set is expanded so that player i can attain weakly higher maximal payoff at each payoff level of other players, then if all else remains equal the new bargaining solution must make individual i weakly better off: $u_i^{*\prime} \geq u_i^*$
R-MON: resource monotonicity	If the bargaining set expands, while the disagreement point is unchanged, nobody should be worse off: $((S \subset S') \wedge (d' = d)) \rightarrow \left(u_i^{*\prime} \geq u_i^*\right), \forall i$
SBS: step-by-step negotiations	The solution is invariant to whether the problem is broken down into steps. First, a subset U of the bargaining set is bargained over, with the original d as the disagreement point. The parties reach an intermediate solution. Then, the parties bargain over the whole bargaining set S, using the intermediate solution as the disagreement point. The final solution they reach is the same as if they had not broken the problem down into steps: $f(S, d) = f(S, f(U, d)), U \subset S$

Table 1: Commonly used bargaining axioms

2.3 Bargaining solutions

A bargaining solution is a function which maps the bargaining problem onto a point in the bargaining set $u^* = f(S, d)$. This paper focuses on three standard cooperative bargaining solutions, Nash, Kalai-Smorodinsky and Egalitarian (or Equal Gains), as well as the asymmetric extension of the latter, the non-Egalitarian Proportional solution, which enables me to illustrate the role of the SYM assumption in relation to MRDU.

2.3.1 Nash solution

Nash (1950) introduced what has become the most widely used cooperative bargaining solution, known as the Nash solution. This is the unique solution satisfying the axioms IR, PAR, SYM, INV and IIA. The Nash solution is defined as the feasible, individually rational and Pareto efficient point u^N maximising the product of players' gains over their disagreement utilities:

$$u^N = \arg \max_{u \in S, u_i > d_i} \prod_j (u_j - d_j)$$

Lemma 1. *Under a linear Pareto frontier, the Nash solution simplifies to:*

$$u_i^N = d_i + \frac{R}{n\eta_i}$$

Or, equivalently:

$$u_i^N = d_i + \frac{T - \sum_j \eta_j d_j}{n\eta_i}$$

Proof. The first order conditions for the Nash optimisation problem subject to the constraint that $\sum_j \left(\eta_j u_j^N\right) = T$, required for Pareto efficiency under a linear Pareto frontier, take the form:

$$\prod_{j \neq i} \left(u_j^N - d_j\right) - \lambda \eta_i = 0$$

Where λ is a Lagrange multiplier. Dividing two of these conditions yields:

$$\eta_i \left(u_i^N - d_i\right) = \eta_k \left(u_k^N - d_k\right)$$

Summing over k yields:

$$n\eta_i\left(u_i^N - d_i\right) = \sum_k \eta_k u_k^N - \sum_k \eta_k d_k$$

Using the fact that $\sum_k \left(\eta_k u_k^N\right) = T$ and rearranging, we obtain:

$$u_i^N = d_i + \frac{T - \sum_k \eta_k d_k}{n\eta_i}$$

□

2.3.2 Kalai-Smorodinsky solution

Kalai and Smorodinsky (1975) axiomatised a popular alternative to the Nash solution, known as the Kalai-Smorodinsky solution. This is the unique solution satisfying IR, PAR, INV, SYM and I-MON. For each player, their ideal utility u_i^+ is characterised as the highest payoff in the individually rational subset of the bargaining set. The point where each player receives their ideal utility is the ideal point u^+(which is likely not feasible itself).

The Kalai-Smorodinsky solution is the feasible, individually rational and Pareto efficient point u^{KS} such that all players have the same ratio between their gain over their disagreement utility and their maximal potential gain over their disagreement utility:

$$\frac{u_i^{KS} - d_i}{u_i^+ - d_i} = x, \forall i : \left(u^{KS} \in S, \neg\exists x \in S : \left(x \geq u^{KS}\right) \wedge \left(\exists i : x_i > u_i^{KS}\right)\right)$$

This solution can be found graphically by the intersection of the utility possibility frontier and the straight line connecting the disagreement point and the ideal point.

Lemma 2. *When the Pareto frontier is linear, the Kalai-Smorodinsky solution coincides with the Nash solution:*

$$u_i^{KS} = d_i + \frac{T - \sum_j \eta_j d_j}{n\eta_i}$$

Proof. Under the constraint that $\sum_j \left(\eta_j u_j^{KS}\right) = T$, required for Pareto efficiency under a linear Pareto frontier, ideal utility is:

$$u_i^+ = \frac{T - \sum_{j \neq i} \eta_j d_j}{\eta_i}$$

Hence, the Kalai-Smorodinsky solution takes the form:

$$\frac{u_i^{KS} - d_i}{\frac{T - \sum_j \eta_j d_j}{\eta_i}} = x$$

Rearranging, summing over i, and simplifying:

$$x = \frac{1}{n}$$

Substituting this into the equation for u_i^{KS}:

$$u_i^{KS} = d_i + \frac{T - \sum_j \eta_j d_j}{n\eta_i}$$

□

2.3.3 Proportional and Egalitarian solutions

Kalai (1977) proposed a class of Proportional solutions, of which the Egalitarian one is a special case. The class of Proportional solutions can be axiomatised in different ways. Kalai (1977) assumes PAR, IR, CSA-INV, and additionally either (i) R-MON, (ii) SBS, or (iii) IIA and I-MON. The Egalitarian solution is the unique solution in this class which additionally satisfies the SYM axiom.

A specific Proportional bargaining solution is defined by a choice of a proportionality vector $\rho > 0$. For any such choice, the Proportional solution u^{PP} is the feasible, individually rational and Pareto efficient point which allocates net gains (over the disagreement point) to players in the ratio determined by ρ:

$$u_i^{PP} = d_i + \rho_i \delta, \sum_j \rho_j = 1, \delta > 0 : \left(u^{PP} \in S, \neg\exists x \in S : \left(x \geq u^{PP}\right) \wedge \left(\exists i : x_i > u_i^{PP}\right)\right)$$

Since the ρ vector and δ can be jointly normalised, I focus on the normalisation $\sum_j \rho_j = 1$, though the definition is equivalent for other normalisations. Note that the ρ vector is a choice that is fixed if, say, the disagreement point changes. Instead δ is a function of the bargaining set and disagreement point.

The Egalitarian solution u^{EG} is given by equal proportionality weights on all parties. Since the ρ vector and δ can be jointly normalised, for this solution I adopt the normalisation $\rho = 1$:

$$u_i^{EG} = d_i + \delta$$

In the two-player case, the Egalitarian solution is found by drawing a 45° line from the disagreement point to the utility possibility frontier.

Lemma 3. *Under a linear Pareto frontier, the Proportional solution simplifies to:*

$$u_i^{PP} = d_i + \rho_i \frac{R}{\sum_j \eta_j \rho_j}$$

Or, equivalently:

$$u_i^{PP} = d_i + \rho_i \frac{T - \sum_j \eta_j d_j}{\sum_j \eta_j \rho_j}$$

Proof. Multiplying $u_j^{PP} = d_j + \rho_j \delta$ through by η_j and summing over j, we obtain:

$$\sum \eta_j u_j^{PP} = \sum \eta_j d_j + \delta \sum \eta_j \rho_j$$

Using the fact that $\sum_j \left(\eta_j u_j^{PP}\right) = T$, required for Pareto efficiency under a linear Pareto frontier, and rearranging:

$$\delta = \frac{T - \sum \eta_j d_j}{\sum \eta_j \rho_j}$$

Substituting this into the equation for the solution:

$$u_i^{PP} = d_i + \rho_i \frac{T - \sum \eta_j d_j}{\sum \eta_j \rho_j}$$

□

2.4 MRDU

MRDU is a particularly intuitive requirement in a two-player context, and several of the examples discussed in section 1 are based on negotiations between two parties.

Definition 1. In a two-player context, MRDU requires that:

$$(\alpha_i > \alpha_{-i}) \to (f_i(S, d + \alpha) > f_i(S, d))$$

Lemma 4. *For any solution which satisfies PAR, in a two-player context MRDU also entails that:*

$$(\alpha_i > \alpha_{-i}) \to (f_{-i}(S, d + \alpha) < f_{-i}(S, d))$$

Definition 2. In the more general n-player context, MRDU requires that:

$$(\alpha_i > \alpha_j, \forall j \neq i) \to (f_i(S, d + \alpha) > f_i(S, d))$$

Lemma 5. *For any solution which satisfies PAR, in an n-player context MRDU also entails that:*

$$(\alpha_i > \alpha_j, \forall j \neq i) \to (\exists j \neq i : f_j(S, d + \alpha) < f_j(S, d))$$

3 The class of bargaining solutions that guarantee MRDU

Under a general bargaining set, the Egalitarian solution is the unique bargaining solution satisfying weak IR which guarantees MRDU.

Lemma 6. *Let $F(S, d)$ be a bargaining solution which satisfies weak IR, and y be a Pareto efficient point. Define an admissible disagreement point $d^l \in S$ which converges to y. Then:*

$$\lim_{d^l \to y} F\left(S, d^l\right) = y$$

Proof. Define:

$$z = \lim_{d^l \to y} F\left(S, d^l\right).$$

By weak IR, $F\left(S, d^l\right) \geq d^l$. Taking limits, $z \geq y$. By the Pareto efficiency of y, it follows that $z = y$. □

Theorem 1. *The Egalitarian solution is the only bargaining solution that satisfies weak IR and MRDU.*

Proof. Let the solution to the bargaining problem defined by disagreement point d and bargaining set S be: $x = F(S, d)$. Let the solution F satisfy weak IR and MRDU. Define the Egalitarian bargaining solution over the same bargaining problem:

$$e = u^{EG}(S, d) = d + \delta$$

Select an arbitrary individual i and define the new disagreement point:

$$d_i^+ = e_i - \epsilon, d_j^+ = e_j - 2\epsilon, \forall j \neq i$$

Where $0 < \varepsilon < \frac{\delta}{2}$ so that $d < d^+ < e$ and hence by comprehensiveness $d^+ \in S$ and the new point is an admissible disagreement point. Individual i has the largest increase in disagreement point:

$$d_i^+ - d_i = \delta - \epsilon, d_j^+ - d_j = \delta - 2\epsilon, \forall j \neq i$$

$$d_i^+ - d_i > d_j^+ - d_j, \forall j \neq i$$

By MRDU,

$$F_i\left(S, d^+\right) > F_i(S, d) = x_i$$

By lemma 6, since the point e is Pareto efficient (Kalai (1977)) and d^+ is an admissible disagreement point which converges to e:

$$\lim_{\epsilon \to 0} d^+ = e$$

It follows that:

$$\lim_{\epsilon \to 0} F\left(S, d^+\right) = e$$

Combining this with $F_i(S, d^+) > x_i$, it follows that $e_i \geq x_i$.

Now define another disagreement point:

$$d_i^- = e_i - 2\epsilon, d_j^- = e_j - \epsilon, \forall j \neq i$$

Where $0 < \varepsilon < \frac{\delta}{2}$ so that $d < d^- < e$ and hence by comprehensiveness $d^- \in S$ and this point is an admissible disagreement point. When the disagreement point changes from d^- to d, individual i has the smallest fall in disagreement utility:

$$d_i - d_i^- = -\delta + 2\epsilon, d_j - d_j^- = -\delta + \epsilon, \forall j \neq i$$

$$d_i - d_i^- > d_j - d_j^-, \forall j \neq i$$

Hence, by MRDU,

$$x_i = F_i (S, d) > F_i \left(S, d^-\right)$$

By lemma 6, since the point e is Pareto efficient (Kalai (1977)) and d^- is an admissible disagreement point which converges to e:

$$\lim_{\epsilon \to 0} d^- = e$$

It follows that:

$$\lim_{\epsilon \to 0} F \left(S, d^-\right) = e$$

Combining this with $x_i > F_i (S, d^-)$, it follows that $x_i \geq e_i$.

Together with $e_i \geq x_i$, this implies that $x_i = e_i$. Since i was chosen arbitrarily, it follows that $x = e$, i.e.

$$F (S, d) = u^{EG} (S, d)$$

□

Therefore, in addition to the characterisation by Kalai (1977), the Egalitarian solution can also be derived by weak IR, the most basic assumption in axiomatic approaches, and MRDU. As well as being the unique solution that satisfies MRDU under weak IR and a general bargaining set, the Egalitarian solution also satisfies other commonly required assumptions, PAR, SYM and CSA-INV, and has other desirable properties such as R-MON, SBS, IIA, I-MON, strong disagreement point monotonicity and strong transfer responsiveness.

3.1 The class of bargaining solutions that guarantee MRDU under TU

Under TU, the Nash and Kalai-Smorodinsky solutions guarantee MRDU even though they do not do so in general. This is because, in this context, they coincide with the Egalitarian solution. More generally, under TU, any solution that satisfies the basic axioms weak IR, PAR, SYM and CSA-INV coincides with the Egalitarian solution and hence satisfies MRDU.[6]

Proposition 1. *Under TU, any solution which satisfies weak IR, PAR, SYM and CSA-INV coincides with the Egalitarian solution.*

Proof. By weak IR and PAR, under TU the solution must satisfy:

$$u_i^* = d_i + \mu_i R$$

where the vector μ may be a function of the bargaining problem, here wholly defined by the disagreement point and the 'size of the pie':

$$\mu_i(d, T) \in [0, 1]$$

Now apply utility transformations in the form $u_i' = au_i + b_i, a > 0$. Under a TU bargaining set, this results in:

$$T' = \sum_j u_j' = a \sum_j u_j + \sum_j b_j = aT + \sum_j b_j$$

$$\sum_j d_j' = a \sum_j d_j + \sum_j b_j$$

$$R' = T' - \sum_j d_j' = aR$$

The new solution must satisfy:

$$u_i^{*\prime} = d_i' + \mu_i' R' = ad_i + b_i + \mu_i' aR$$

[6] The Nash and Kalai-Smorodinsky solutions satisfy INV, which is stronger than CSA-INV and hence implies CSA-INV. In fact, the TU requirement forces these two assumptions to coincide.

CSA-INV requires that:

$$u_i^{*\prime} = au_i^* + b_i = ad_i + a\mu_i R + b_i$$

These conditions jointly imply $\mu_i' = \mu_i$, i.e. that the μ vector is a constant:

$$\mu(d, T) = c$$

Finally, by SYM, under TU if $d_i = x, \forall i$ then $u_i^* = \frac{T}{n}, \forall i$. This implies that:

$$x + c_i R = \frac{T}{n}$$

Noting that $R = T - \sum_j d_j$, this entails:

$$c_i (T - nx) = \frac{1}{n} (T - nx)$$

With solution:

$$c_i = \frac{1}{n}, \forall i$$

Hence the unique bargaining solution satisfying weak IR, PAR, SYM and CSA-INV under TU is:

$$u_i^* = d_i + \frac{R}{n}$$

This solution equalises gains relative to disagreement for players, and is hence equivalent to the Egalitarian solution. □

Theorem 2. *Under TU, the class of solutions which satisfies weak IR, PAR, SYM and CSA-INV coincides with the class of solutions which satisfies weak IR and MRDU.*

Proof. By theorem 1, the Egalitarian solution is the only solution which satisfies both weak IR and MRDU. By proposition 1, under TU, any solution that satisfies weak IR, PAR, SYM and CSA-INV coincides with the Egalitarian solution. Hence, under TU, any solution that satisfies weak IR, PAR, SYM and CSA-INV also satisfies MRDU. Moreover, any solution that satisfies MRDU must coincide with the Egalitarian solution, which satisfies weak IR, PAR, SYM and CSA-INV. Therefore, the class of solutions which satisfies both weak IR and MRDU is the class of solutions satisfying weak IR, PAR, SYM and CSA-INV. □

Corollary 1. *The Nash solution and the Kalai-Smorodinsky solution satisfy MRDU under TU.*

Proof. Both the Nash and Kalai-Smorodinsky solutions satisfy weak IR, PAR, SYM and CSA-INV (they satisfy IR, which implies weak IR, and INV, which implies CSA-INV) and hence by theorem 2 satisfy MRDU under TU. □

Corollary 2. *The non-Egalitarian Proportional solution does not satisfy MRDU under TU.*

Proof. The non-Egalitarian Proportional solution does not satisfy SYM and hence by theorem 2 it does not satisfy MRDU under TU. □

4 Nash solution

The Nash solution does not guarantee MRDU under a general (non-TU) bargaining set. It is useful to understand in what cases the Nash solution violates MRDU.

Example 1. Let the initial disagreement point be the origin and the bargaining set be:

$$S = \left\{(u_1, u_2) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq (1 - u_1)^{1/5}\right\}$$

The Nash solution yields $(0.833, 0.699)$, rounded to 3 d.p. If we change the disagreement point to $d = (0.2, 0.19)$, this yields Nash solution $(0.828, 0.703)$, to 3 d.p. The bargaining outcome improves for player 2 and worsens for player 1 even though the disagreement utility improved more for player 1 than for player 2.

4.1 The Nash solution does not guarantee MRDU under a non-TU linear Pareto frontier

Proposition 2. *If the bargaining set is linear but not characterised by TU then the Nash solution does not guarantee MRDU.*

Proof. When the disagreement point changes by a vector $\alpha > 0$, by lemma 1, this results in the following change in outcome for player i:

$$\Delta_i^N = \alpha_i - \frac{\sum_j \eta_j \alpha_j}{n \eta_i}$$

Because we have ruled out TU, there must exist a low utility weight individual i:

$$\exists i : \eta_i < \frac{1}{n}$$

Let this low utility weight individual experience the largest increase in disagreement utility:

$$\alpha_i = \sigma + \epsilon, \alpha_k = \sigma, \forall k \neq i, \sigma, \epsilon > 0$$

Then:

$$\Delta_i^N = \sigma \left(1 - \frac{1}{n\eta_i}\right) + \epsilon \left(1 - \frac{1}{n}\right)$$

The second term must be positive. Because $\sigma > 0$ and because we have chosen a low utility weight individual, the first term must be negative. By selecting suitable σ and ϵ, such that the former is large relative to the latter, while both are small enough that the disagreement point remains feasible, we can construct an MRDU violation: $\Delta_i^N < 0$ even though $\alpha_i > \alpha_k, \forall k \neq i$. □

Lemma 7. *Under the Nash bargaining solution, with a non-TU linear Pareto frontier it is always possible to construct counterexamples to MRDU where (i) all disagreement utilities increased, or where (ii) all disagreement utilities decreased.*

Proof. Part (i) is proven by the proof of proposition 2. The same framework is straightforwardly adapted to prove part (ii). In this case, let i be a high utility weight individual, for whom $\eta_i > \frac{1}{n}$, and let them experience the smallest decrease in disagreement utility:

$$\alpha_i = \sigma + \epsilon, \alpha_k = \sigma, \forall k \neq i, \sigma < 0, \epsilon > 0, \alpha_i < 0$$

In the equation for Δ_i^N the second term is again positive. Because $\sigma < 0$ and because i is a high utility weight individual, the first term is again negative. Again, a counterexample to MRDU can be constructed by selecting suitable large $|\sigma|$ relative to ϵ. □

The Nash solution violates MRDU under a non-TU linear Pareto frontier when all disagreement utilities increased (decreased) if and only if: (i) the player with the largest increase (smallest decrease) in disagreement utility has a sufficiently low (high) utility weight, and (ii) inter-player differences in disagreement utilities changes are sufficiently small.

Lemma 8. *With a non-TU linear Pareto frontier, let exactly $x < n-1$ players $k \in \Omega$ experience a decrease in disagreement utility, and all other players $j \notin \Omega$ experience an increase in disagreement utility. The Nash solution can violate MRDU if and only if there exists a very low utility weight individual:*

$$\exists i : n\eta_i < \sum_{j \neq \Omega} \eta_j$$

Under a non-TU linear Pareto frontier, the Nash solution guarantees MRDU if only one player's disagreement utility increases and all others decrease.

Proof. Write:

$$\sum_j \eta_j \alpha_j = \sum_{j \notin \Omega} \eta_j \alpha_j + \sum_{k \in \Omega} \eta_k \alpha_k$$

Let i be the player with the maximal increase in disagreement utility:

$$\alpha_i = \max_j \alpha_j$$

Then:

$$\sum_{j \notin \Omega} \eta_j \alpha_j < \alpha_i \sum_{j \notin \Omega} \eta_j$$

And:

$$\sum_{k \in \Omega} \eta_k \alpha_k < 0$$

Hence:

$$\sum_j \eta_j \alpha_j < \alpha_i \sum_{j \notin \Omega} \eta_j$$

For all three inequalities, the LHS can approach the RHS from below with suitable choice of α.

The outcome of individual i worsens if $\Delta_i^N < 0$, which occurs when:

$$n\eta_i \alpha_i < \sum_j \eta_j \alpha_j$$

This is possible if and only if i is a very low weight individual:

$$n\eta_i < \sum_{j \notin \Omega} \eta_j$$

The existence of a very low weight individual is not guaranteed by the restriction that the Pareto frontier is non-TU linear. Where no such individual exists, the Nash solution cannot violate MRDU.

Moreover, when $x = n - 1$ the condition requires that $n\eta_i < \eta_i$, which is impossible.

Where a very low weight individual exists, an MRDU violation can be constructed by selecting a suitably small ϵ and letting:

$$\alpha_i = \sigma, \alpha_j = \sigma - \epsilon, \forall j \neq i, j \notin \Omega, \alpha_k = -\epsilon, \forall k \in \Omega, \sigma, \epsilon > 0$$

□

The Nash solution violates MRDU under a non-TU linear Pareto frontier when some disagreement utilities increased and others decreased if and only if: (i) the disagreement utility of a player with sufficiently low utility weight increases most, (ii) the utility weights of other players whose disagreement utility increases are sufficiently high, (iii) the number of players who experience falls in disagreement utility $x < n-1$ is sufficiently low, and (iv) inter-player differences in disagreement utilities changes are sufficiently small.

4.1.1 Full characterisation of violations in the two-player case

I turn to the two-player case because this is often the most empirically relevant, and it provides the results that I build on in section 7 to compare the nature and likelihood of MRDU violations under different bargaining solutions.

Proposition 3. *With a non-TU linear Pareto frontier and two players, the Nash solution violates MRDU if and only if one of the following alternatives holds:*

$$\frac{\eta_i}{\eta_{-i}} < 1, \alpha_i > 0, \alpha_{-i} > 0, \frac{\eta_i}{\eta_{-i}}\alpha_i < \alpha_{-i} < \alpha_i$$

$$\frac{\eta_i}{\eta_{-i}} < 1, \alpha_i < 0, \alpha_{-i} < 0, \alpha_i < \alpha_{-i} < \frac{\eta_i}{\eta_{-i}}\alpha_i$$

Proof. In the two-player case:

$$\Delta_i^N = \frac{\eta_i \alpha_i - \eta_{-i}\alpha_{-i}}{2\eta_i}$$

The utility of i falls when:

$$\eta_i \alpha_i < \eta_{-i}\alpha_{-i}$$

This contradicts MRDU when $\alpha_i > \alpha_{-i}$. Combining the inequalities yields:

$$\frac{\eta_i}{\eta_{-i}}\alpha_i < \alpha_{-i} < \alpha_i$$

Unless $\eta_1 = \eta_2$ (the TU case), this double inequality can hold for suitable choices of α_i and α_{-i}. For the double inequality to hold we require either that both α_i and α_{-i} are positive, or that they are both negative. $\square$

Corollary 3. *In the two-player case, with a linear Pareto frontier, the Nash solution guarantees MRDU where one player's disagreement utility increases and the other decreases (i.e. $\alpha_1 > 0$ and $\alpha_2 < 0$, or vice versa).*

If the disagreement point improves (worsens) for both players, but more significantly for one of them, then this leads to an MRDU violation only if the improvement (worsening) for this player relative to the other was smaller than the ratio between utility weights.

4.2 The Nash solution guarantees MRDU only under TU

Lemma 9. *If a bargaining solution does not guarantee MRDU under any (non-TU) linear Pareto frontier then it follows that the bargaining solution does not guarantee MRDU under any (non-TU) Pareto frontier.*

Proof. For any bargaining set as defined in section 2.1, it must be possible to approximate at least a, potentially infinitesimally small, part of any (non-TU) Pareto frontier as a (non-TU) linear frontier. Therefore, it is possible to construct an example of an MRDU violation which is local to that segment of the Pareto frontier. This can be done by selecting an initial disagreement point marginally lower than the point around which the set is being linearly approximated, so that the latter is the initial bargaining outcome. Then a violation can be constructed by a suitable and infinitesimally small change in the disagreement point. □

Proposition 4. *The Nash solution guarantees MRDU only under TU.*

Proof. This is jointly implied by corollary 1, proposition 2 and lemma 9. □

4.3 Discussion of the conditions under which the Nash solution guarantees MRDU

Intuitively, MRDU is a sensible requirement only if interpersonal utility comparisons are allowed. Hence, the core reason why the Nash bargaining solution does not in general guarantee MRDU is that it is not grounded in a framework where interpersonal utility comparisons are meaningful. The INV assumption embedded in the Nash bargaining solution means that we can scale different player's utilities by different amounts, rendering it meaningless to say that one player's disagreement utility has increased more than another's.

If we instead restrict our bargaining set to TU then that effectively invalidates the possibility of rescaling utilities of different players by different amounts, so that the INV assumption reduces to the weaker CSA-INV assumption. Therefore, the Nash solution becomes consistent with interpersonal utility comparisons of utility gains.

5 Kalai-Smorodinsky solution

The Kalai-Smorodinsky solution does not guarantee MRDU under a general (non-TU) bargaining set. It is useful to understand in what cases the Kalai-Smorodinsky solution violates MRDU.

Example 2. Let the initial disagreement point be the origin and the bargaining set be:

$$S = \left\{(u_1, u_2) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq (1 - u_1)^{1/5}\right\}$$

The Kalai-Smorodinsky solution is $(0.755, 0.755)$,rounded to 3 d.p. Let the disagreement utility increase for player 2 more than for player 1, to $(0.45, 0.5)$. The Kalai-Smorodinsky outcome shifts to $(0.775, 0.742)$, rounded to 3 d.p., violating MRDU.

This is because, plotting u_2 on the y axis, the Pareto frontier is steeper at its x-intercept, u_1^+, than its y-intercept, u_2^+. Therefore, a slightly larger increase in disagreement utility for player 2 than player 1 translates into a more-than-proportionately larger fall in ideal utility for player 2 than for player 1 (from 1 to 0.887 for the former, from 1 to 0.969 for the latter). The combination of these changes results in a worse outcome for player 2.

5.1 The Kalai-Smorodinsky solution guarantees MRDU only under TU

Proposition 5. *If the bargaining set is linear but not characterised by TU then the Kalai-Smorodinsky solution does not satisfy MRDU.*

Proof. By lemma 2, the Kalai-Smorodinsky solution is identical to the Nash solution under a linear Pareto frontier. Hence, the proof provided for proposition 2 also applies to proposition 5. □

The other results obtained under the Nash solution in the non-TU linear Pareto frontier setting, lemma 7 and lemma 8, and specifically in the two-player case, proposition 3 and corollary 3, extend to the Kalai-Smorodinsky solution.

Proposition 6. *The Kalai-Smorodinsky solution guarantees MRDU only under TU.*

Proof. This is jointly implied by corollary 1, proposition 5 and lemma 9. □

5.2 Discussion of the conditions under which the Kalai-Smorodinsky solution guarantees MRDU

Mechanically, the Kalai-Smorodinsky solution does not guarantee MRDU because a change in disagreement point also entails a change in the ideal point. This happens because the disagreement point affects which subset of the bargaining set is individually rational. When *only* one player's disagreement utility improves, their ideal utility remains unchanged. Conversely, the other player's disagreement utility is fixed, but their ideal utility worsens. These changes lead to an improved outcome for the player with the improved disagreement utility, as showed by Thomson (1987).

However, when the disagreement point moves for both players at the same time, the ideal utility also shifts for both players. How much a player's ideal utility shifts depends both on the magnitude of the change in the other player's disagreement utility, and on the shape of the utility possibility frontier. If the slope of the frontier is locally very steep close to the ideal utility, then even a small change in the other player's outside option will substantially change one's own ideal utility. Depending on the relative local slopes of the frontier close to the ideal utilities of each player, even if the disagreement point improves in relative terms for one player, the ideal utility may as a consequence worsen in relative terms for that same player. When the latter effect outweighs the former this leads the player with the relative improvement in outside option to have a worsened Kalai-Smorodinsky bargaining outcome.

As discussed in the case of the Nash bargaining solution, MRDU is a sensible requirement only if interpersonal comparisons of utility gains are allowed. Hence, the core reason why the Kalai-Smorodinsky solution does not in general satisfy MRDU is that it is not grounded in a framework where interpersonal utility comparisons are meaningful due to the INV assumption. If we instead restrict our bargaining set to TU then the Kalai-Smorodinsky solution becomes consistent with cardinal unit comparability.

6 Non-Egalitarian Proportional solution

The non-Egalitarian Proportional solution does not guarantee MRDU under a general bargaining set. It is useful to understand in what cases the non-Egalitarian Proportional solution violates MRDU.

Example 3. Let $\rho_1 = \frac{1}{3}$, $\rho_2 = \frac{2}{3}$, the initial disagreement point be the origin and the bargaining set be:

$$S = \left\{(u_1, u_2) : 0 \leq u_1 \leq 1, 0 \leq u_2 \leq (1 - u_1)^{1/5}\right\}$$

The initial solution is $(0.445, 0.889)$, rounded to 3 d.p. Let the disagreement point change to $(0.4, 0.5)$ .The new solution is $(0.572, 0.844)$, rounded to 3 d.p. Player 2's payoff fell even though their disagreement utility increased more than player 1's.

A player with a higher (lower) proportionality weight can lose out from an improvement (worsening) in disagreement utility even if they have the larger increase (smaller fall) in disagreement utility because of the resulting decrease (increase) in surplus to be shared, of which they receive a larger (smaller) share. As long as the difference between their own gain (loss) in disagreement utility and the other party's is small relative to the asymmetry in the way the surplus is split between parties, this can lead to a loss. An extreme example of an improvement in disagreement utility is when the old solution is no longer individually rational for the player with the lower proportionality weight, so that their outcome must improve even if they had a smaller increase in disagreement utility.

6.1 The non-Egalitarian Proportional solution does not guarantee MRDU under a linear Pareto frontier

Proposition 7. *If the bargaining set is linear (regardless of whether it is characterised by TU) then the non-Egalitarian Proportional solution does not guarantee MRDU.*

Proof. Let i be a high proportionality weight individual, for whom $\rho_i > \sum_j \eta_j \rho_j$. Such an individual must exist as long as the solution is non-Egalitarian. By lemma 3, the impact on party i of the disagreement point changing by vector α is:

$$\Delta_i^{PP} = \alpha_i - \rho_i \frac{\sum_j \eta_j \alpha_j}{\sum_j \eta_j \rho_j}$$

Let the high proportionality weight individual experience the largest increase in disagreement utility:

$$\alpha_i = \sigma + \epsilon, \alpha_k = \sigma, \forall k \neq i, \sigma, \epsilon > 0$$

Then:

$$\Delta_i^{PP} = \sigma \left(1 - \frac{\rho_i}{\sum_j \eta_j \rho_j}\right) + \epsilon \left(1 - \eta_i\right)$$

The second term must be positive. Because $\sigma > 0$ and because we have chosen a high proportionality weight individual, the first term must be negative. By selecting suitable σ and ϵ, such that the former is large relative to the latter, while both are small enough that the disagreement point remains feasible, we can construct an MRDU violation: $\Delta_i^{PP} < 0$ even though $\alpha_i > \alpha_k, \forall k \neq i$. □

Lemma 10. *Under the non-Egalitarian Proportional solution, with a linear Pareto frontier (regardless of whether it is characterised by TU) it is always possible to construct counterexamples to MRDU where (i) all disagreement utilities increased, or where (ii) all disagreement utilities decreased.*

Proof. Part (i) is proven by the proof of proposition 7. The same framework is straightforwardly adapted to prove part (ii). In this case, let i be a low proportionality weight individual, for whom $\rho_i < \sum_j \eta_j \rho_j$, and let them experience the smallest decrease in disagreement utility:

$$\alpha_i = \sigma + \epsilon, \alpha_k = \sigma, \forall k \neq i, \sigma < 0, \epsilon > 0, \alpha_i < 0$$

In the equation for Δ_i^{PP} the second term is again positive. Because $\sigma < 0$ and because i is a low proportionality weight individual, the first term is again negative. Again, a counterexample to MRDU can be constructed by selecting suitable large $|\sigma|$ relative to ϵ. □

The non-Egalitarian Proportional solution violates MRDU under a linear Pareto frontier when all disagreement utilities increased (decreased) if and only if: (i) the player with the largest increase (smallest decrease) in disagreement utility has a sufficiently high (low) proportionality weight, and (ii) inter-player differences in disagreement utilities changes are sufficiently small.

Lemma 11. *With a linear Pareto frontier (regardless of whether it is characterised by TU), let exactly $x < n - 1$ players $k \in \Omega$ experience a decrease in disagreement utility, and all other players $j \notin \Omega$ experience an increase in disagreement utility. The non-Egalitarian Proportional solution can violate MRDU if and only if there exists a very high proportionality weight individual:*

$$\rho_i \sum_{j \neq \Omega} \eta_j > \sum_j \eta_j \rho_j$$

Under a linear Pareto frontier, the non-Egalitarian Proportional solution guarantees MRDU if only one player's disagreement utility increases and all others decrease.

Proof. Write:

$$\sum_j \eta_j \alpha_j = \sum_{j \notin \Omega} \eta_j \alpha_j + \sum_{k \in \Omega} \eta_k \alpha_k$$

Let i be the player with the maximal increase in disagreement utility:

$$\alpha_i = \max_j \alpha_j$$

Then:

$$\sum_{j \notin \Omega} \eta_j \alpha_j < \alpha_i \sum_{j \notin \Omega} \eta_j$$

And:

$$\sum_{k \in \Omega} \eta_k \alpha_k < 0$$

Hence:

$$\sum_j \eta_j \alpha_j < \alpha_i \sum_{j \notin \Omega} \eta_j$$

For all three inequalities, the LHS can approach the RHS from below with suitable choice of α.

The outcome of individual i worsens if $\Delta_i^N < 0$, which occurs when:

$$\sum_j \eta_j \alpha_j > \frac{\alpha_i \sum_j \eta_j \rho_j}{\rho_i}$$

This is possible if and only if i is a very high proportionality weight individual:

$$\rho_i \sum_{j \notin \Omega} \eta_j > \sum_j \eta_j \rho_j$$

The existence of a very high proportionality weight individual is not guaranteed by the restriction that the solution is non-Egalitarian. Where no such individual exists, the Proportional solution cannot violate MRDU.

Moreover, when $x = n - 1$ the condition requires that $\rho_i \eta_i > \sum_j \eta_j \rho_j$, which is impossible.

Where the condition is met, an MRDU violation can be constructed by selecting a suitably small ϵ and letting:

$$\alpha_i = \sigma, \alpha_j = \sigma - \epsilon, \forall j \neq i, j \notin \Omega, \alpha_k = -\epsilon, \forall k \in \Omega, \sigma, \epsilon > 0$$

□

The non-Egalitarian Proportional solution violates MRDU under a linear Pareto frontier when some disagreement utilities increased and others decreased if and only if: (i) the disagreement utility of a player with sufficiently high proportionality weight increases most, (ii) the proportionality weights of other players whose disagreement utility increases are sufficiently low, (iii) the utility weights and proportionality weights of players whose disagreement utility decreases are sufficiently low, (iv) the number of players who experience falls in disagreement utility $x < n - 1$ is sufficiently low, and (v) inter-player differences in disagreement utilities changes are sufficiently small.

6.1.1 Full characterisation of violations in the two-player case

I turn to the two-player case because this is often the most empirically relevant, and it provides the results that I build on in section 7 to compare the nature and likelihood of MRDU violations under different bargaining solutions.

Proposition 8. *With a linear Pareto frontier and two players, the non-Egalitarian Proportional solution violates MRDU if and only if one of the following alternatives holds:*

$$\rho_i > \rho_{-i}, \alpha_i > 0, \alpha_{-i} > 0, \frac{\rho_{-i}}{\rho_i}\alpha_i < \alpha_{-i} < \alpha_i$$

$$\rho_i > \rho_{-i}, \alpha_i < 0, \alpha_{-i} < 0, \alpha_i < \alpha_{-i} < \frac{\rho_{-i}}{\rho_i}\alpha_i$$

Proof. In the two-player case:

$$\Delta_i^{PP} = \alpha_i - \frac{\rho_i}{\eta_i\rho_i + \eta_{-i}\rho_{-i}}\left(\eta_i\alpha_i + \eta_{-i}\alpha_{-i}\right)$$

The utility of i falls when:

$$\alpha_{-i} > \frac{\rho_{-i}}{\rho_i}\alpha_i$$

This contradicts MRDU when $\alpha_i > \alpha_{-i}$. Combining the inequalities yields:

$$\frac{\rho_{-i}}{\rho_i}\alpha_i < \alpha_{-i} < \alpha_i$$

Unless $\rho_1 = \rho_2$ (the Egalitarian case), this double inequality can hold for suitable choices of α_i and α_{-i}. For the double inequality to hold we require either that both α_i and α_{-i} are positive, or that they are both negative. □

Corollary 4. *In the two-player case, with a linear Pareto frontier, the non-Egalitarian Proportional solution guarantees MRDU where one player's disagreement utility increases and the other decreases (i.e. $\alpha_1 > 0$ and $\alpha_2 < 0$, or vice versa).*

If the disagreement point improves (worsens) for both players, but more significantly for one of them, then this leads to an MRDU violation only if the improvement (worsening) for this player relative to the other was smaller than the ratio between proportionality weights.

6.2 The non-Egalitarian Proportional solution does not guarantee MRDU under any bargaining set restriction

Proposition 9. *The non-Egalitarian Proportional solution does not guarantee MRDU under any bargaining set restriction.*

Proof. This is jointly implied by proposition 5 and lemma 9. □

Intuitively, requiring the MRDU property implicitly assumes a symmetric treatment of players. A symmetric treatment is implied by the idea that if a player's disagreement utility increases more than the other player's (not by a specific factor higher than 1) then their outcome should improve. Since the non-Egalitarian Proportional solution does not satisfy the SYM assumption, this solution can violate MRDU regardless of the shape of the bargaining set.

7 Characterisation of MRDU violations

While the Egalitarian solution guarantees MRDU, the Nash, Kalai-Smorodinsky and non-Egalitarian Proportional solutions sometimes violate it. It is useful to understand how common such violations are, and characterise the disagreement point changes which give rise to them. In this section, I characterise the nature and share of possible changes in the disagreement point that lead to MRDU violations for different bargaining solutions. To this end, I consider exclusively a two-player setting with a non-TU linear bargaining frontier. I use 'Nash / K-S' as a shorthand for the Nash and Kalai-Smorodinsky solutions, since these coincide under a linear bargaining frontier. I also use 'Proportional' as a short-hand for 'non-Egalitarian Proportional'. Without loss of generality, let $\eta_i < \eta_{-i}$.

I illustrate examples of the share of possible disagreement point changes contradicting MRDU for the Nash / K-S solution in figure 1 and for the Proportional solution in figure 2. Recall that, for both the Nash / K-S solution and the non-Egalitarian Proportional solution, MRDU violations in the two-player case can occur only when the disagreement point improves for all players, or worsens for all players (corollary 3 and corollary 4). These figures can be interpreted either for an improvement or a worsening of the disagreement point. In the case of an improvement, let the origin be the original disagreement point and let the new disagreement point fall somewhere in the bargaining set. The shaded area are the points that would give rise to an MRDU contradiction, while any other points would lead to a change consistent with MRDU. If we are instead considering a worsening of the disagreement point, let the origin be the new disagreement point. Let the old disagreement point fall somewhere in the bargaining set. Again, the shaded area are the points that would give rise to an MRDU contradiction, while any other points would lead to a change consistent with MRDU.

Findings are reported in table 2. The nature of violations column summarises the inequalities from propositions 3 and 8. The share column gives the area of the subset which leads to MRDU violations relative to the area of the bargaining set, as a function of parameters. The bounds column provides the lower and upper bound for the share given the range of possible values that parameters can take.

To discuss the bounds, I focus on the case of an improvement in disagreement point. For the Nash / K-S solution, violations of MRDU happen when the player with a lower utility weight experiences a larger increase in disagreement utility than the other player, but the difference is small relative to the ratio of utility weights. As the utility weights approach equality, we approach the TU case, and the share of possible changes in disagreement point which produce MRDU violations goes to 0. As the higher utility weight approaches unity, and the lower one zero, we approach the case where an MRDU violation occurs whenever the player with a lower utility weight experiences a larger increase in disagreement utility than the other player, i.e. half of possible scenarios.

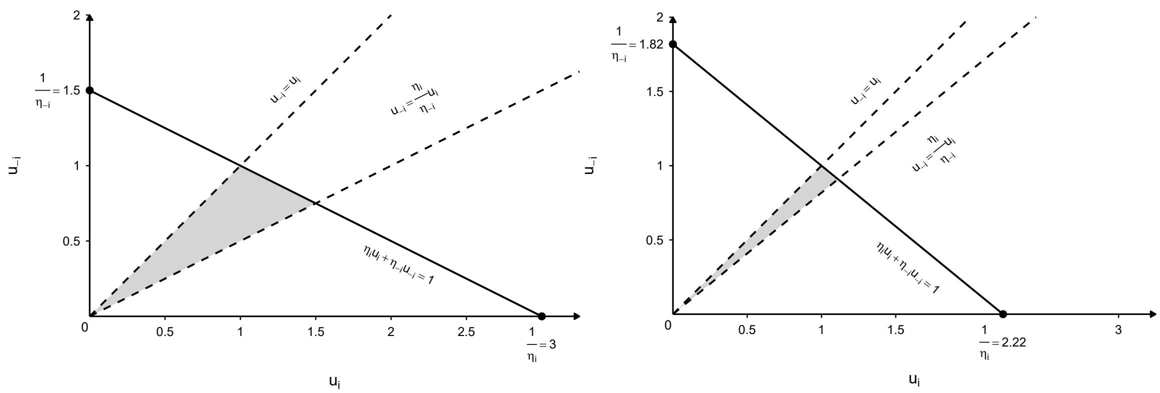

Figure 1: Share of possible disagreement point changes which violate MRDU under the Nash / K-S solution. Illustrated with values of $\eta_i = \frac{1}{3}$ in the left panel and $\eta_i = 0.45$ in the right panel. T normalised to 1.

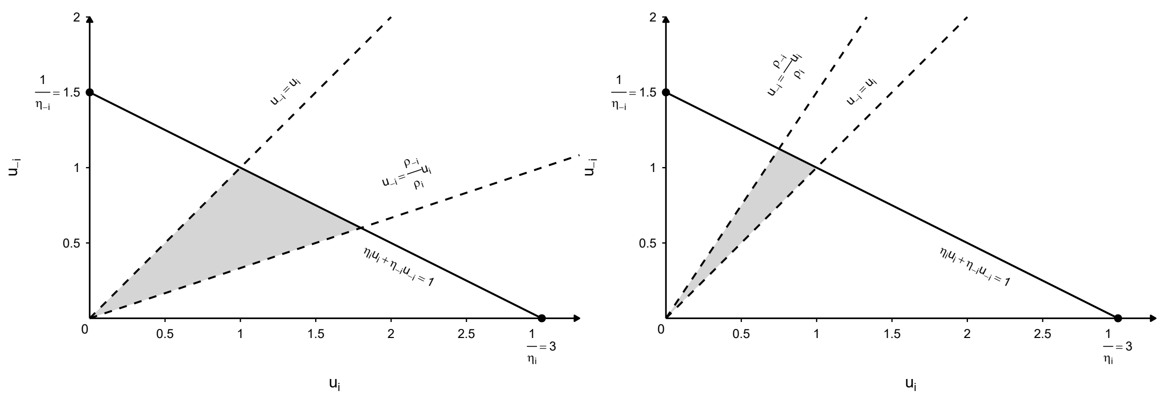

Figure 2: Share of possible disagreement point changes which violate MRDU under the Proportional solution. Illustrated with $\eta_i = \frac{1}{3}$, and values of $\rho_i = 0.75$ in the left panel and $\rho_i = 0.4$ in the right panel. T normalised to 1.

For the Proportional solution, we must distinguish between the case where $\rho_i > \rho_{-i}$ (the player with lower utility weight has higher proportionality weight) and the case where $\rho_i < \rho_{-i}$ (the player with lower utility weight has lower proportionality weight).

In the first case, the type of changes which give rise to MRDU violations is qualitatively similar to those which give rise to MRDU violations under the Nash / K-S solution. These happen when the player with a lower utility weight, and higher proportionality weight, experiences a larger increase in disagreement utility than the other player, but the difference is small relative to the ratio between proportionality weights. When the proportionality weights approach equality (Egalitarian case) the share of violations goes to 0. When the higher proportionality weight approaches unity, and the lower one zero, the share of violations goes to η_{-i}. Since $\eta_i < \eta_{-i}$, it follows that $\eta_{-i} \in (0.5, 1)$. Hence the share of violations can fall anywhere within the unit interval.

In the second case, where $\rho_i < \rho_{-i}$, the type of changes which give rise to MRDU violations is qualitatively different from those which give rise to MRDU violations under the Nash / K-S solution. These happen when the player with higher proportionality weight, and *higher* utility weight, experiences a larger increase in disagreement utility than the other player, but the difference is small relative to the ratio between proportionality weights. Again, when the proportionality weights approach equality (Egalitarian case) the share of violations goes to 0. When the higher proportionality weight approaches unity, and the lower one zero, the share of violations goes to η_i. Since $\eta_i < \eta_{-i}$, it follows that $\eta_i \in (0, 0.5)$.

Solution	Nature of violations	Share	Bounds
Nash / K-S	$\frac{\eta_i}{\eta_{-i}} \lvert\alpha_i\rvert < \lvert\alpha_{-i}\rvert < \lvert\alpha_i\rvert$	$\frac{\eta_{-i}-\eta_i}{2}$	$(0, 0.5)$
Egalitarian	NA	0	0
Proportional $\rho_i > \rho_{-i}$	$\frac{\rho_{-i}}{\rho_i} \lvert\alpha_i\rvert < \lvert\alpha_{-i}\rvert < \lvert\alpha_i\rvert$	$\frac{(\rho_i-\rho_{-i})\eta_i\eta_{-i}}{\eta_i\rho_i+\eta_{-i}\rho_{-i}}$	$(0, 1)$
Proportional $\rho_i < \rho_{-i}$	$\lvert\alpha_i\rvert < \lvert\alpha_{-i}\rvert < \frac{\rho_{-i}}{\rho_i} \lvert\alpha_i\rvert$	$\frac{(\rho_{-i}-\rho_i)\eta_i\eta_{-i}}{\eta_i\rho_i+\eta_{-i}\rho_{-i}}$	$(0, 0.5)$

Table 2: MRDU violations with linear Pareto frontier, $n = 2, \eta_i < \eta_{-i}$. Note that violations are possible only when either $\alpha_1, \alpha_2 > 0$ or $\alpha_1, \alpha_2 < 0$.

For a general Proportional solution with unspecified ρ, the share of changes which leads to an MRDU violation is not consistently larger or smaller for the Nash / K-S solution than for the Proportional solution, so that it is not possible to say with confidence that one of these solutions is “closer” to satisfying MRDU in general. However, what can be said is that the upper bound for the Proportional solution is twice as high as for the Nash / K-S solution. Moreover, for $\frac{\rho_{-i}}{\rho_i} = \frac{\eta_i}{\eta_{-i}}$ the subset of possible disagreement point changes that contradict MRDU coincide for the Nash / K-S and Proportional solutions.

Continuing to focus on the case of the disagreement point improving for both players, I briefly discuss the scenarios in which (i) neither Nash / K-S nor Proportional solutions violate MRDU, (ii) both Nash / K-S and Proportional solutions violate MRDU, and (iii) one of the Nash / K-S and Proportional solutions violates MRDU while the other does not. First, there will be no MRDU violations if the difference between disagreement utility increases was large relative to both utility and proportionality weight ratios. Moreover, there cannot be an MRDU violation when a player with both high utility weight and low proportionality weight had the larger increase in disagreement utility. Second, if the player with the larger increase had both a low utility weight and a high proportionality weight, and the difference was small, relative to both the utility and proportionality weight ratios, then both the Nash / K-S and Proportional solutions lead to MRDU violations. Third, where the difference between disagreement utility increases is small relative to the utility but not the proportionality weight, or vice versa, or where the person with low utility weight has a low proportionality weight, the Nash / K-S solution may give rise to MRDU violations where the Proportional solution does not, and vice versa.

8 Discussion

This paper defines and studies MRDU as a property of cooperative bargaining solutions. MRDU is intuitively appealing, relevant to answering interesting applied questions, implicitly assumed in contexts including academic economics and antitrust cases, and appears to have some empirical validity, though it has never been formally tested.

I show that, among the set of bargaining solutions satisfying weak IR, only the Egalitarian solution guarantees MRDU. This result adds to an existing literature which has found the Egalitarian solution to satisfy other desirable properties that are not satisfied by other standard solutions, including the more commonly used Nash solution

(Thomson (1987), Bossert (1994)).

While, for a general bargaining set, MRDU is guaranteed only by the Egalitarian solution, I show that, where TU is an acceptable assumption, MRDU is guaranteed by any solution which satisfies the axioms weak IR, PAR, CSA-INV and SYM. This includes the Nash and Kalai-Smorodinsky solutions as well as the Egalitarian solution. Where TU is considered too unrealistic, the likelihood of MRDU violations depends on the specific bargaining solution, the specific disagreement point change, and the shape of the bargaining set. All else equal, the larger the difference between the most favourable, and other, disagreement utility changes, the less likely an MRDU violation.

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